MAT137 - Applications

- Today's lecture will assume you have watched videos 14.11, 14.12, 4.13 14.14

  No videos for tomorrow!

- Please take a few minutes to fill out your course evaluation! They really do matter.

Lagrange's Remainder Theorem

Reminder of the theorem from the video:

Theorem

Let f be C^{n+1} on an interval I containing a point a.

Then for any $x \in I$, we have:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c in between a and x.

This is a consequence of the MVT.
Note that the value of c depends on both n and x.

The easiest way to use this theorem is to put an upper bound M on $\left|f^{(n+1)}(c)\right|$, so you don't have to care about the particular c.

The sine function is analytic

In this exercise you'll prove that $f(x) = \sin(x)$ is analytic on $\mathbb{R}$.

1. (A warm up, just to set a goal.) Fix a real number a, and write down the Taylor series for $\sin(x)$ centred at a.

2. Fix an $x \in \mathbb{R}$ and a non-negative integer n, and use Lagrange's theorem to write down an expression for the remainder $R_n(x)$.
(Remember to quantify your variables.)

3. Find a positive number M such that $|f^{(n+1)}(c)| < M$, no matter what c is.
(Hint: This is easy and you definitely know how to do it already.)

4. Prove that $\lim_{n\to\infty} |R_n(x)| = 0$.

5. Prove that $\lim_{n\to\infty} R_n(x) = 0$.

We conclude that $\sin(x)$ is analytic on all of $\mathbb{R}$!

Integrals

Problem: We want to compute the value of

$$A = \int_0^1 x^{17} \sin(x) \, dx.$$

There are two ways you can do this:

1. Integrate by parts 17 times to find an antiderivative.
2. Use power series to find an antiderivative.

Use whichever one you think is faster.

Follow-up problem. Estimate the value of A with an error smaller than 0.001. (At least convince yourself of how to do this.)

Add these series

1. $A = \sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$ *Hint:* Think of sin
2. $B = \sum_{n=0}^{\infty} (4n+1)x^{4n+2}$ *Hint:* $\frac{d}{dx}\left[x^{4n+1}\right] = ???$
3. $C = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$ *Hint:* Write the first few terms. Combine e^x and e^{-x}
4. $D = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$ *Hint:* Integrate

Computing Maclaurin Series

Problem. Find the **first four non-zero terms** of the Maclaurin series of these functions:

1. $f(x) = e^x \sin x$
2. $g(x) = e^{\sin x}$

Hint: Treat the power series the same way you would treat a polynomial.

L'Hopital's Rule Sucks 1

We just found out that:

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

Problem 1. What is $g^{(4)}(0)$?

Problem 2. What is $\lim_{x \to 0} \frac{e^{\sin x} - 1 - x}{x^2 + x^3}$?

Problem 3. What is $\lim_{x \to 0} \frac{e^{\sin x} - 1 - x}{\cos x - 1}$?

Problem 4. What is $\lim_{x \to 0} \frac{x^2(e^{\sin x} - 1 - x)}{(\cos x - 1)^2}$?

L'Hopital's Rule Sucks 2

Use Maclaurin series to compute these limits:

1. $\displaystyle \lim_{x \to 0} \frac{6 \sin x - 6x + x^3}{x^5}$

2. $\displaystyle \lim_{x \to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4}$

3. $\displaystyle \lim_{x \to 0} \frac{(\sin x - x)^3 x}{(\cos x - 1)^4 (e^x - 1)^2}$