MAT137 - Week 8 Lecture 2

- Today's lecture will assume you have watched videos 3.13, 3.15, 3.16, 3.17, 3.18.
  For Monday's lecture, watch videos 3.19, 3.20, 4.1, 4.2

Implicit Differentiation

Consider some relation between x and y which defines a curve on the plane.
Recall that if the tangent line at some point (x_0, y_0) on the curve is not vertical, then we can locally express y as a function of x.

We have shown this for the circle! Specifically we have shown that for any point on the circle that is not $(\pm 1, 0)$,

If $y > 0$, then

$$y = \sqrt{1 - x^2}$$

If $y < 0$, then

$$y = -\sqrt{1 - x^2}$$

From there, we found that the slope of the tangent at any point on the circle (except $(\pm 1, 0)$) is:

$$\frac{dy}{dx} = -\frac{x}{y}$$

whenever the right side makes sense. (so when $y \neq 0$).

Implicit Differentiation

Another method (called **implicit differentiation**) to find $\frac{dy}{dx}$ is by assuming for the moment that y is a function of x. And just take the derivative of both sides with respect to x (use the chain rule):

$$\frac{d}{dx}\left[x^2 + y^2 = 1\right]$$

We get

$$2x + 2yy' = 0$$

which gives us the same thing as before:

$$y' = -\frac{x}{y}$$

(Implicit differentiation give us a method to find $\frac{dy}{dx}$ without needing to express y as an explicit function of x.)

Implicit differentiation

Consider the relation

$$\sin(x + y) + xy^2 = 0$$

graph It is very difficult to solve for y and express y as a function of x and take the derivative to get the slope of the tangent. So Implicit Differentiation is the only tool that always works.

This relation defines a function $y = h(x)$ near $(0, 0)$. Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$

Compute the derivative of the following functions:

1. $f(x) = e^{\sin x + \cos x} \ln x$
2. $f(x) = \pi^{\tan x}$
3. $f(x) = \ln \left[e^x + \ln \ln \ln x \right]$

Reminder: We know:

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} a^x = a^x \ln a$
- $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$

Logarithmic differentiation.

Problem. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.

2. $g(x) = x^{\tan(x)}$.

3. Now generalize these ideas into a new differentiation rule:

Let f and g be differentiable functions, and define h by

$$h(x) = [f(x)]^{g(x)}\,.$$

Derive a formula for $h'(x)$.

More logarithmic differentiation

Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\begin{aligned}
\ln f(x) =& (\cos x)\ln(\sin x) + (\sin x)(\ln \cos x) \\
\frac{d}{dx}[\ln f(x)] =& \frac{d}{dx}[(\cos x)\ln(\sin x)] + \frac{d}{dx}[(\sin x)(\ln \cos x)] \\
\frac{f'(x)}{f(x)} =& -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x} \\
& + (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}
\end{aligned}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

Do it correctly as an exercise

A different type of logarithm

Do this as an exercise

Calculate the derivative of

$$f(x) = \log_{x+1}(x^2 + 1)$$

Hint: If you do not know where to start, remember the definition of logarithm:

$$\log_a b = c \iff a^c = b.$$

Hard derivatives made easy.

With the tools you now know, you can more or less differentiate any function you can right down.

For example, you can compute the derivative of:

$$h(x) = \sqrt[3]{\frac{(\sin^6 x)\sqrt{x^7 + 6x + 2}}{3^x (x^{10} + 2x)^{10}}}$$

It will be long, but easy. Taking a log of both sides will turn the right side into a long sum, which is easy to differentiate.