MAT137 - Week 8 Lecture 1

- Today's lecture will assume you have watched videos 3.10, 3.11, 3.12. **For Tuesday's lecture, watch videos 3.13, 3.15, 3.16, 3.17, 3.18**

Derivatives of $(f \circ g)$

Assume f and g are functions that have all their derivatives.
Find formulas for

1. $(f \circ g)'(x)$
2. $(f \circ g)''(x)$
3. $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

Chain Rule + Product Rule $\implies$ Quotient Rule

Problem 1. Let $c \in \mathbb{R}$, and let g be a function which is differentiable at c and such that $g(c) \neq 0$.

Let $h(x) = \dfrac{1}{g(x)}$.

Use the chain rule to derive a formula for $h'(c)$.

Problem 2. Use your formula from above to give a simple proof of the quotient rule.

Derivative of cos

In one of the videos, you saw a derivation of the derivative of sin.

Let $g(x) = \cos x$.

Derive a formula for its derivative directly from the definition of the derivative as a limit.

Hint: Use the "$h \to 0$" version of the definition of the derivative, and imitate the derivation in Video 3.11.

Hint: This identity may come in handy:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

Trigonometric derivatives

In order to prove:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x \tag{1}$$

we had to use the following facts:

1. $\lim_{x \to 0} \frac{\sin x}{x}$.
2. $\lim_{x \to 0} \frac{\cos x - 1}{x}$

which we have proven geometrically.

Now using (1), give simple proofs of those facts.

Trigonometric derivatives

Do this as an exercise

You now know that:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$

Problem. Evaluate the following limits:

1. $\lim_{h \to 0} \frac{\cos(7(x+h)) - \cos(7x)}{h}$.
2. $\lim_{h \to 0} \frac{\sin(7x+h) - \sin(7x)}{h}$

Trigonometric derivatives

Do this as an exercise

Again, you now know that:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$

From these two results, quickly derive the formulas for the derivatives of...

1. $\tan(x)$
2. $\sec(x)$
3. $\csc(x)$
4. $\cot(x)$

A pesky function

Let

$$h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Is h continuous at 0?
4. Is h differentiable at 0?
5. Is h' continuous at 0?

Remark: This is an example of a function f that is differentiable everywhere, but its derivative f' is not continuous everywhere. In fact, $\lim_{x \to 0} f'(x)$ does not even exist.

Answer the same questions for the functions $x \sin \dfrac{1}{x}$ and $x^3 \sin \dfrac{1}{x}$

Implicit Differentiation

Function: For each input there is a unique output. $x \longmapsto f(x)$

We can define the graph of the function as the set $\Big\{\Big(x, f(x)\Big) : x \in \mathbb{R}\Big\}$.

(This will be a curve on the plane).

Relation: A relationship between several variables with no well-defined idea of an input and an output, in particular no "uniqueness" of output. The set $\Big\{(x, y) : x \text{ and } y \text{ satisfy some relation together } \Big\}$ is also a curve.

Example: Consider the relation $x^2 + y^2 = 1$. Then the set is $\Big\{(x, y) : x^2 + y^2 = 1\Big\}$ is the circle centered at 0 with radius 1. Can you think of a function such that its graph is that curve?

At every point in this curve, is there a well defined tangent line? YES!

However, saying something like $\frac{d}{dx}\Big|_{x=a} y$ will not always make sense. Why?

Implicit Differentiation

There is a deep theorem that says any curve with a well defined tangent line at every point can *locally* be written as the graph of some function.

In particular, if the tangent line at some point (x_0, y_0) on the curve is not vertical, then there exists a small enough rectangle centered at (x_0, y_0) such that y can be *uniquely* written as a function of x inside that rectangle. This is called the **implicit function theorem**. Let us believe it for now.

Prove it for the circle! And find $\frac{dy}{dx}$ at the points where the tangent line is not vertical.
So you got that:

$$\frac{dy}{dx} = -\frac{x}{y}$$

For any (x, y) on the circle except $(\pm 1, 0)$