MAT137 - Week 11 Lecture 1

- **Reminder:** Reminder: Test 2 is on Friday 29 November. Please read the vocabulary list posted on the course website before the test.
- Problem Set B is on the website now. It contains material that is not covered by Problem Sets 1 through 4, but that is covered by Test 2. It is not to be submitted, but it is very good practise. Do these problems before Test 2.
- Today's lecture will assume you have watched videos 5.7, 5.8, 5.9, 5.10, 5.11, 5.12

  For Tuesday's lecture, watch videos 6.1, 6.2

Using the MVT

Problem 1: Let f be a function that is continuous on $[0, 7]$ and differentiable on $(0, 7)$.
Suppose that $f(0) = -5$, and that $f'(x) \leq 10$ for all $x \in (0, 7)$.

What can you say about the value of $f(7)$?

Problem 2: Let $a, b \in \mathbb{R}$. Use the Mean Value Theorem to show

$$|\cos a - \cos b| \leq |a - b|$$

Problem 3: We can generalize problem 2. Prove this theorem:

Theorem

Let f be differentiable on $\mathbb{R}$ and suppose f' is bounded. Then there exists $M > 0$ such that:

$$\forall x, y \in \mathbb{R}, \;\; |f(x) - f(y)| \leq M|x - y|$$

Positive derivative implies increasing

We want to prove that if a function has positive derivative on an interval, then it's increasing on that interval. We begin with a warm up:

Let f be a function defined on an interval I.

Problem 1: Write the definition of "f is increasing on I".

Problem 2: Write down the statement of the Mean Value Theorem. (Don't omit the hypotheses).

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. From the statement, figure out the structure of the proof.
2. If you have used a theorem, did you verify the hypotheses?
3. Are there words in your proof, or just equations?

What is wrong with this proof?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$, so $f(b) > f(a)$
- f is increasing.

□

Proving inequalities

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: When is the function $f(x) = e^x - 1 - x$ increasing or decreasing?

is this theorem true?

Theorem

Let $a < b$ and let f be a function defined on $[a, b]$.
IF

- f is differentiable on (a, b)
- $\forall x \in (a, b)$, $f'(x) = 0$,

Then f is constant on $[a, b]$

If you think it's true, then prove it.
If you think it's false, come up with a counter example.

Proving difficult identities

Prove that there exists a $C \in \mathbb{R}$ such that for every $x \geq 0$,

$$\arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x} = C$$

Hint: Let $g(x) = \arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x}$. I am asking you to prove that g is constant on $[0, \infty)$

What's wrong with this "proof"?

Proof.

- g is differentiable on $(0, \infty)$.
- Then for any $x \in (0, \infty)$

$$g'(x) = \frac{d}{dx}\left[\arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right)\right] = \ldots$$

- ...(derivative computations)...
- $g'(x) = 0$ on $(0, \infty)$.
- Therefore, by the MVT, g is constant on $(0, \infty)$.
- We evaluate it at $x = 0$ to find the constant value.
- $g(0) = \ldots\text{(computations)}\ldots = \frac{\pi}{2}$
- Therefore, $g(x) = \frac{\pi}{2}$ for all $x \geq 0$.

Challenging Question

Is this theorem true or false?

Theorem

Let f be differentiable on $\mathbb{R}$. Then f' cannot have jump or removable discontinuities.

If you think it's true, prove it.
If you think it's false, find a counter example.

Challenging Question

Theorem

Let f be differentiable on $\mathbb{R}$. Then f' cannot have jump or removable discontinuities.

Let's first prove this lemma.

Lemma

Let f be differentiable on $\mathbb{R}$.
IF

- $\lim_{x \to a^+} f'(a)$ exists,
- $\lim_{x \to a^-} f'(a)$ exists,

THEN $\lim_{x \to a} f'(x) = f'(a)$