MAT137 - More Power Series

- Today's lecture will assume you have watched videos 14.9, 14.10

  For Tuesday's lecture, watch videos 14.11, 14.12, 4.13 14.14

Computing Taylor series

Write the following functions as power series centered at 0:

1. $g(x) = \ln(1 + x)$
2. $f(x) = \sin^2 x$.
3. $f(x) = \arctan x$

Hint: For each of the first two, take one single derivative. Then stop to think.

For the third one, *do not* try to multiply power series together. (It's possible, but harder than necessary here.)

Taylor series gymnastics

Problem 1. Write the following functions as power series centered at 0. You don't need to compute any derivatives to do these.

1. $f(x) = \dfrac{x^2}{1+x}$
2. $f(x) = x^5 \log(1+x^3)$
3. $f(x) = \sin\left(2x^3\right)$
4. $f(x) = \ln \dfrac{1+x}{1-x}$
5. $f(x) = e^x(1-x^2)$
6. $f(x) = \dfrac{1}{(1+x^2)(1+x)}$

Problem 2. For each function in previous question, compute $f^{(2019)}(0)$.

$f(x) = \arctan x$ is analytic at 0

We will prove that $f(x) = \arctan x$ is analytic at 0.

If you are allowed to switch the order of the sum and the integral, then this is very easy (done in slide 2). Let us try to do it without using that fact

1. Find the n^{th} Taylor polynomial for f at 0 (Do not use Method 1 from slide 8). Find an expression for the remainder $R_n(x)$ in integral form.

2. Find the Taylor series for f at 0. Where does it converge?

3. Show f agrees with its Taylor series on an interval centred at 0. Conclude that f is analytic at 0.

Lagrange's Remainder Theorem

Reminder of the theorem from the video:

Theorem

Let f be C^{n+1} on an interval I containing a point a.

Then for any $x \in I$, we have:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c in between a and x.

This is a consequence of the MVT.
Note that the value of c depends on both n and x.

The easiest way to use this theorem is to put an upper bound M on $\left|f^{(n+1)}(c)\right|$, so you don't have to care about the particular c.

The sine function is analytic

In this exercise you'll prove that $f(x) = \sin(x)$ is analytic on $\mathbb{R}$.

1. (A warm up, just to set a goal.) Fix a real number a, and write down the Taylor series for $\sin(x)$ centred at a.

2. Fix an $x \in \mathbb{R}$ and a non-negative integer n, and use Lagrange's theorem to write down an expression for the remainder $R_n(x)$.
(Remember to quantify your variables.)

3. Find a positive number M such that $|f^{(n+1)}(c)| < M$, no matter what c is.
(Hint: This is easy and you definitely know how to do it already.)

4. Prove that $\lim\limits_{n\to\infty} |R_n(x)| = 0$.

5. Prove that $\lim\limits_{n\to\infty} R_n(x) = 0$.

We conclude that $\sin(x)$ is analytic on all of $\mathbb{R}$!