MAT137 - Week 3 Lecture 1

- **Reminder:** Problem Set 1 is available on the course website, and is due Thursday, September 26 by 11:59pm.

- Today's lecture will assume you have watched videos 2.1 - 2.4
  For Tuesday's lecture, watch videos 2.5 and 2.6.

Problem 1. Let a, b, c be real numbers. Assume that $a < b$. Which of the following must then be true?

1. $a + c < b + c$.
2. $a - c < b - c$.
3. $ac < bc$.
4. $a^2 < b^2$.
5. $\frac{1}{a} < \frac{1}{b}$.

Absolute values and inequalities

Problem. For what values of x is the following inequality true?

$$|x - 7| < 3$$

In other words, what values of x are within a distance 3 from 7?

Notice that thinking about the expression $|x - 7|$ as "the distance between x and 7" makes this problem *much* easier.

What about the following inequality?

$$0 < |x - 7| < 3$$

Some review of absolute values and inequalities

Problem. Suppose x is a real number that satisfies the inequality

$$|x-2| < 1.$$

What bounds, if any, can you put on $|x-7|$?

Here's how you should read this question:

If x is within a distance 1 from 2, how far can x be from 7?

You have just proved the following conditional:

$$|x-2| < 1 \implies 4 < |x-7| < 6$$

Is the other direction true?

Some review of absolute values and inequalities

Now we know how to bound values of x with inqualities and absolute values. Let's use bounds on x to find bounds on the values of functions of x.

Problem. Suppose x is a real number that satisfies the inequality

$$|x+7| < 2.$$

How big can $|3x+21|$ be?

In words:

If x is within a distance 2 from -7, how far can $3x$ be from -21?

You have just proved the following conditional:

$$|x+7| < 2 \implies |3x+21| < 6$$

Absolute values and conditionals

Now let's reverse idea in the previous exercise.

Problem 1. Find **one** positive number δ that makes the following conditional true.

$$\text{If } |x-3|<\delta, \text{ then } |4x-12|<6.$$

In words:

> *How close should x be to 3 so that $4x$ is within a distance 6 from 12?*

Problem 2. Find **all** positive numbers δ that make the above conditional true.

Absolute values and conditionals

Now we know that for any $0 < \delta \leq \frac{3}{2}$, the following conditional is true:

$$\text{If } |x - 3| < \delta, \text{ then } |4x - 12| < 6.$$

We'll work with this idea a bit.

Problem 3. Suppose we want a tighter restriction on $|4x - 12|$ in the conditional above. For example, let's say we want the distance between $4x$ and 12 to be less than 1:

$$\text{If } |x - 3| < \delta, \text{ then } |4x - 12| < 1.$$

Will all of the same values of δ that worked before work now?

No! To make $4x$ closer to 12, we must make x closer to 3. Which values of δ will work here?

Absolute values and conditionals

Problem 4. Let ϵ be a fixed positive real number. Is it possible to find a positive δ that **does not** depend on ϵ and that makes the following conditional true?

$$\text{If } |x-3| < \delta, \text{ then } |4x-12| < \epsilon.$$

No! The smaller ϵ is, the smaller δ should have to be! Just like before.

Problem 5. Let ϵ be a fixed positive real number. Find a value of δ, in terms of ϵ, that makes the following conditional true:

$$\text{If } |x-3| < \delta, \text{ then } |4x-12| < \epsilon.$$

Absolute values and conditionals

Now we know that for a fixed positive real number ϵ, the following conditional is true:

$$\text{If } |x-3| < \frac{\epsilon}{4}, \text{ then } |4x-12| < \epsilon.$$

Congratulations! You've just done the "hard part" of proving that $\lim_{x \to 3} 4x = 12$.

Limits, intuitively

Recall the intuitive definition of a limit given in the videos:

$$\lim_{x \to c} f(x) = L \qquad \text{means}$$

If x is close to c (but not equal to c), then $f(x)$ is close to L.

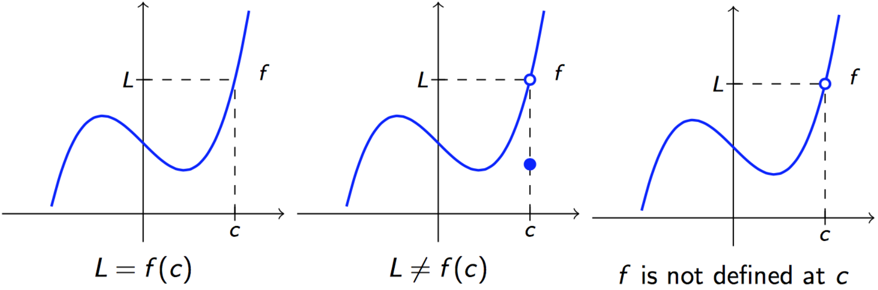
Slightly more precisely:

For every interval centred at L, there exists a small enough interval centred at c that gets mapped by f into the interval centred at L.

Limits, intuitively

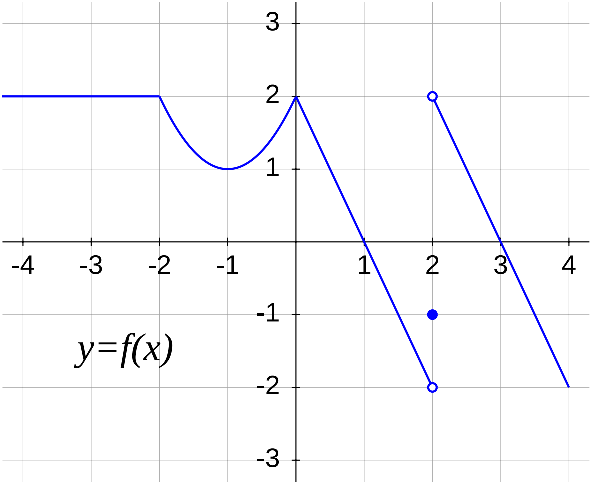
Note that a limit ***never*** cares about what's happening at c. Only near c.

All of the following functions have the same limit L at c:



In particular, this means that in principle you can ***never*** evaluate a limit simply by plugging $x = c$ into the function.

Limits from a graph



Find the value of

1. $\lim_{x \to 2} f(x)$
2. $\lim_{x \to 0} f(f(x))$
3. $\lim_{x \to -3} f(f(x))$
4. $\lim_{x \to 2} [f(x)]^2$