MAT137 - Taylor Series, Analytic functions

- Today's lecture will assume you have watched videos 14.5, 14.6, 14.7, 14.8

  For Tuesday's lecture, watch videos 14.9, 14.10

An explicit equation for Taylor polynomials

1. Find one polynomial P of degree 3 that satisfies

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Let f be a C^3 function. Find an explicit formula for the 3-rd Taylor polynomial for a function f at 0.

4. Let f be a C^∞ function, and let $n > 0$. Find an explicit formula for the n-th Taylor polynomial for a function f at 0.

Warm up: Recall the Definitions

Taylor Polynomial

Let f be C^n at a. The n^{th} Taylor polynomial P_n for f about $x = a$ is the *unique* polynomial with the smallest degree satisfying:

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

The explicit formula for P_n is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Problem 1: Define the Taylor series of f at a.

Problem 2: Define what it means for f to be analytic at a.

Taylor polynomial of a polynomial

Let $f(x) = x^3$.

1. Write the n^{th} Taylor polynomial P_n for f at 0
   Then write the Taylor series of f at 0.
   Is f analytic at 0?

2. Write the n^{th} Taylor polynomial P_n for f at 1
   Then write the Taylor series of f at 1.
   Is f analytic at 1?

True or False

Let f be a function defined at least on an interval I centred at a. Are the following true or false?

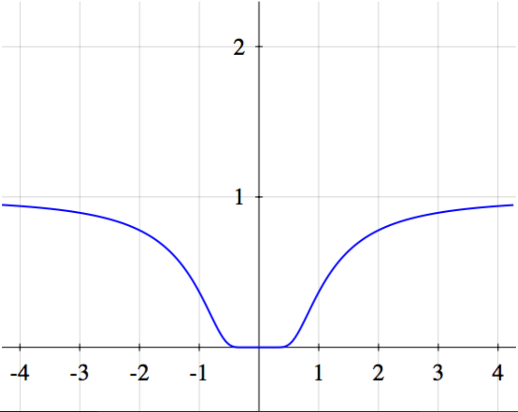
1. If f is analytic on I, then f is C^∞ on I.
2. If f is C^∞ on I, then f is analytic on I
3. If f is C^∞ on I and it's Taylor series at a converges on I, then f is analytic on I.

A counterexample

Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here's the relevant part of its graph:



A counterexample

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is a a tedious (but purely computational) exercise to check that:

$$f^{(k)}(0) = 0 \quad \text{for all } k = 0, 1, 2, \ldots$$

You may assume this without proof.

Exercise:

- What is the Taylor series of this function at 0?
- Where does this series converge absolutely?
- For which points x does the function equal its Taylor series?
- Is f analytic at 0?

Taylor series not at 0

Write the Taylor series...

1. ...for $f(x) = e^x$ at $a = 2$.
2. ...for $g(x) = \sin x$ at $a = \frac{\pi}{4}$.
3. ...for $H(x) = \frac{1}{x}$ at $a = 3$.

You can do these problems in two ways:

- Method 1: Compute the first few derivatives, guess the pattern (and prove it by induction).
- Method 2: Use the substitution $u = x - a$ and reduce it to an old problem (without computing any derivatives).

More Taylor series

Do this as an exercise

Write the following functions as power series centered at 0:

1. $g(x) = \ln(1 + x)$
2. $f(x) = \sin^2 x$.

Hint: For each of the first two, take one single derivative. Then stop to think.

For the third one, *do not* try to multiply power series together. (It's possible, but harder than necessary here.)