- Today's lecture will assume you have watched videos 3.6, 3.7, 3.9.
  For Monday's lecture, watch videos 3.10, 3.11, 3.12
- Do not forget Anonymous Feedback.

When can we use the Quotient rule.

Let f and g be functions defined at and near a. Suppose f and g are differentiable at a. Define a new function h by

$$h(x) = \frac{f(x)}{g(x)}.$$

Is h necessarily differentiable at a? NO! h might not even be defined at a or even near a

What extra conditions do we need to impose on f and g to ensure that h is differentiable at a? We need $g(a) \neq 0$

The following lemma will be used to prove the quotient rule:

Lemma

Let f, g, and h be functions as defined above. If $g(a) \neq 0$, then h is defined on an interval centered at a
(i.e. $g(x) \neq 0$ on an interval centered at a).

Write a formal proof for the quotient rule for derivatives

Theorem

- Let $a \in \mathbb{R}$.
- Let f and g be functions defined at and near a. Assume $g(a) \neq 0$.
- We define the function h by $h(x) = \dfrac{f(x)}{g(x)}$.

IF f and g are differentiable at a,

THEN h is differentiable at a, and

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Write a proof directly from the definition of derivative.
Hint: Imitate the proof of the product rule in Video 3.6.

Recall a trick from the product rule.

In order to prove the product rule, we had to compute a similar limit, and to do that we did a simple "trick" of adding zero in a creative way:

$$\begin{aligned}
&\frac{f(x)g(x) - f(a)g(a)}{x-a} \\
=&\frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\
=&\frac{f(x) - f(a)}{x-a}\, g(x) + f(a)\frac{g(x) - g(a)}{x-a}
\end{aligned}$$

A similar (but not identical) trick will help you with this proof.

Be careful to explicitly justify any limits you evaluate in your proof.

Check your proof of the quotient rule

1. Did you use the *definition* of the derivative?
2. Are there only equations and no words? If so, you haven't written a proof.
3. Does every step follow logically from the previous steps (with explanation)?
4. Did you assume anything you couldn't assume?
5. Did you assume at any point that a function is differentiatiable? If so, did you justify it?
6. Did you assume at any point that a function is continuous? If so, did you justify it?

If you answered "no" to Q6 above, your proof cannot be fully correct.

Critique this proof

$$h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)}$$

$$= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)}$$

$$= \lim_{x\to a} \left\{ \left[\frac{f(x)-f(a)}{x-a} g(a) - f(a)\frac{g(x)-g(a)}{x-a} \right] \frac{1}{g(x)g(a)} \right\}$$

$$= \left[f'(a)g(a) - f(a)g'(a) \right] \frac{1}{g(a)g(a)}$$

Extending the power rule.

In the videos you saw a proof of the power rule for natural exponents, done by induction:

Theorem

Let n be a positive integer, and let f be the function defined by $f(x) = x^n$.

Then f is differentiable everywhere, and

$$f'(x) = nx^{n-1}.$$

Prove this corollary: Do this as an exercise

Corollary

Any polynomial is differentiable everywhere.

Extending the power rule.

Now that we know the quotient rule, we can extend this result to negative exponents.

Prove the following theorem:

Theorem

Let n be a positive integer, and let f be the function defined by $f(x) = x^{-n}$.

Then f is differentiable everywhere except zero, and

$$f'(x) = -nx^{-n-1} = -\frac{n}{x^{n+1}} \quad (x \neq 0)$$

What would need in order to extend this to rational numbers? In other words, if a and b are integers, what would we have to know in order to compute the derivative of $f(x) = x^{a/b}$?