MAT137 - Week 7 Lecture 1

- Today's lecture will assume you have watched videos 3.1 - 3.5, 3.8.
  For Tuesday's lecture, watch videos 3.6, 3.7, 3.9

Some quick problems, to make sure you watched the videos

Problem 1. If we know f at a, then which of the following do we know?

Problem 2. If we know f on an interval centered at a but not a, then which of the following do we know?

Problem 3. If we know f on an interval centered at a, then which of the following do we know?

1. whether $\lim_{x \to a} f(x)$ exists or not.
2. whether f is continuous at a or not.
3. whether f is differentiable at a or not.
4. whether f is differentiable twice at a or not.

This is the same as asking: Can you find two functions f and g such that $f(x) = g(x)$ for all $x \in$ " * " but f satisfies "$\#$" and g satisfies NOT "$\#$".

Level 1: True or False

Let $a \in \mathbb{R}$.

Let f be a function with domain $\mathbb{R}$.

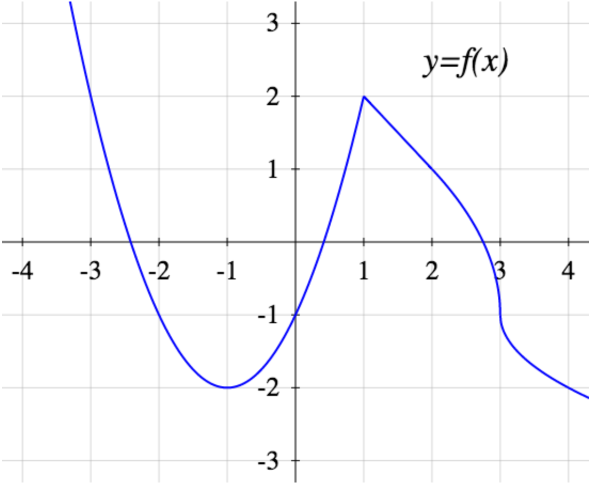
Assume f is differentiable everywhere.

What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a

Sketching the graph of f' from the graph of f

Below is the graph of the function f. Sketch the graph of f'.



Derivatives.

Problem 1. Let f be the function defined by $f(x) = x|x|$.

Is f differentiable at 0? If so, what is its derivative?

Another way to write this function is $f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$

Level 2: True and False

Prove or disprove:

1. If f is differentiable everywhere, and $\lim_{x\to\infty} f'(x) = 0$, then $\lim_{x\to\infty} f(x)$ exists.
2. If $\lim_{x\to a} f'(x) = \infty$, then f is unbounded at any interval centered at a but not a.
3. Let f and g be differentiable functions on $\mathbb{R}$ such that for all x, $f(x) \leq g(x)$. Then $f'(x) \leq g'(x)$ for all x.
4. Let g be a continuous function defined on $\mathbb{R}$. If $\lim_{x\to 0} \frac{g(x)}{x}$ exists then g is differentiable at 0. Solution in the next slide
5. If $f(x) = g(x)$ on an interval centered at a, then f is differentiable at a iff g is differentiable at a. If so, then $f'(a) = g'(a)$.
6. If f is differentiable at a, then $f'(a) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$.

We want to prove:

Theorem

Let g be a continuous function defined on $\mathbb{R}$. If $\lim_{x \to 0} \frac{g(x)}{x}$ exists then g is differentiable at 0.

Since $\lim_{x \to 0} \frac{g(x)}{x}$ and $\lim_{x \to 0} x$ both exist, we can use the limit product rule to conclude the limit of the product exists; so we get:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{g(x)}{x} x = \left(\lim_{x \to 0} \frac{g(x)}{x} \right) \left(\lim_{x \to 0} x \right) = \left(\lim_{x \to 0} \frac{g(x)}{x} \right) 0 = 0$$

Since g is continuous at 0, we also get:

$$g(0) = \lim_{x \to 0} g(x) = 0$$

We need to show that g is differentiable at 0. Note that:

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h)}{h}$$

which exists as given in the question (notice that we used that $g(0) = 0$). So we conclude that g is differentiable at 0 and $g'(0) = \lim_{x \to 0} \frac{g(x)}{x}$.

Higher derivatives

Do this as an exercise

Problem. Let $f(x) = \dfrac{1}{x^7}$.

1. Calculate the first few derivatives of f.
2. Make a conjecture for a formula for the n^{th} derivative of f.
3. Prove your formula. (The easiest way to do this is by induction.)