MAT137 - FTC

- Today's lecture will assume you have watched videos 8.3, 8.4, 8.5, 8.6, 8.7

  For Tuesday's lecture, watch videos 9.1, 9.4

Problem 1: Prove this theorem:

Theorem

Let f and g be integrable functions on $[a, b]$ satisfying $f \geq g$. Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Problem 2: Prove this theorem:

Theorem

Let f be a continuous on $[a, b]$ satisfying $f \geq 0$. Suppose also that $f(c) > 0$ for some $c \in (a, b)$. Then

$$\int_a^b f(x)dx > 0$$

Defining functions with integrals

Let f be an integrable function defined on an interval $[a, b]$. Let c and d be elements of $[a, b]$.

Question: Consider the following expressions.

1. $\int_c^x f(x)\,dx$
2. $\int_c^x f(u)\,du$
3. $\int_c^x f(c)\,dt$
4. $\int_c^x f(t)\,dt$

Which of these expressions make sense? Do any of them equal one another?

FTC warm up

Let f be a continuous function with domain $\mathbb{R}$.

Problem 1: Find one antiderivative of f.

Problem 2: Find all antiderivatives of f.

Problem 3: Find all antiderivatives G of f satisfying $G(5) = 20$.

Remark: We have just shown that if we specify the derivative of a function and choose it's value at one point, then that determines the function uniquely.

Problem 4: What does FTC part 2 say about $\int_a^b f(t)dt$?

True, false, or can't decide?

Do this as an exercise

We want to find a function H with domain $\mathbb{R}$ such that $H(1) = -2$ and such that

$$H'(x) = e^{\sin x} \quad \text{for all } x.$$

Decide whether each of the following statements is true, false, or you do not have enough information to decide.

1. The function $H(x) = \int_0^x e^{\sin t}\, dt$ is a solution.
2. The function $H(x) = \int_2^x e^{\sin t}\, dt$ is a solution.
3. $\forall C \in \mathbb{R}$, the function $H(x) = \int_0^x e^{\sin t}\, dt + C$ is a solution.
4. $\exists C \in \mathbb{R}$ s.t. the function $H(x) = \int_0^x e^{\sin t}\, dt + C$ is a solution.
5. The function $H(x) = \int_1^x e^{\sin t}\, dt - 2$ is a solution.
6. There is more than one solution.

Counter example for FTC?

Let f be a differentiable function on $\mathbb{R}$. Recall that f' cannot have jump discontinuities.

Here is a counter example that uses FTC. What is wrong?

Counter example

Let $g(x) = 2$ when $x \geq 0$ and $g(x) = 0$ when $x < 0$.
Define f in the following way:

$$f(x) = \int_0^x g(t)dt$$

So by the FTC, $f'(x) = g(x)$, but g has a jump discontinuity at $x = 0$.

Hint: Compute f; then compute its derivative.

The first FTC

The following statement of this theorem is a little bit more general than the one given in the video.

Theorem (First Fundamental Theorem of Calculus)

Suppose f is integrable on $[a, b]$, and let $c \in [a, b]$. Then the function

$$F(x) = \int_c^x f(t)\, dt$$

is continuous on $[a, b]$.

F is differentiable at any point x where f is continuous, and at such a point $F'(x) = f(x)$.

Prove it! Do this as an exercise.

Using FTC1

The main thing we use FTC1 for in this course is proving FTC2. But it does have some applications of its own, as a new differentiation rule.

Problem 1: Find the derivative of $F(x) = \int_0^x \cos^2(t)\,dt$.

Problem 2: Find the derivative of $G(x) = \int_x^7 7te^{t^2}\,dt$.

Problem 3: Find the derivative of $H(x) = \int_0^{x^2} \frac{\sin(t)}{t}\,dt$.

Hint: The answer is not $\frac{\sin(x^2)}{x^2}$.

Going further...

Problem 4: Find the derivative of $S(x) = \int_0^x xe^{t^2}\, dt$.

Problem 5: Find the derivative of $F(x) = \int_{x^2}^{\sin(x)} \arctan(t)\, dt$.

Hint: Try to write $F(x)$ as a sum of two functions whose derivatives you know how to compute.

Okay! We're [trying to be] mathematicians, so it's time to generalize! Why solve one example, when we can solve **all** the examples?

Exercise

Let f, u, v be differentiable functions on $\mathbb{R}$, and define:

$$F(x) = \int_{u(x)}^{v(x)} f(t)\, dt.$$

Find a formula for $F'(x)$ in terms of some or all of f, u, v, f', u', v'.

An optimization problem. (Remember those?)

For each real number $a > 0$, consider the function

$$f_a(x) = 1 + a - ax^2.$$

At this point, stop and draw a rough sketch of what the graph of such a function looks like.

Find the value of a that minimizes the area of the region bounded by the graph of f_a and the x-axis.