MAT137 - Week 10 Lecture 1

- **Reminder:** Problem Set 4 is due Thursday 21 November, by 11:59pm.

- Today's lecture will assume you have watched videos 4.6, 4.7, 4.8, 5.1, 5.2, 5.3, 5.4

  For Tuesday's lecture, watch videos 5.5, 5.6

Left and right inverses

Let $f : A \to B$

Problem 1: Suppose there exists a function $g : B \to A$ such that $\forall y \in B, f(g(y)) = y$. Show that f is surjective.

Problem 2: Suppose there exists a function $g : B \to A$ such that $\forall x \in A, g(f(x)) = x$. Show that f is injective.

Remark: If there exists a function $g : B \to A$ such that both the above are satisfied, then f is invertible and g is the inverse.

The goal today

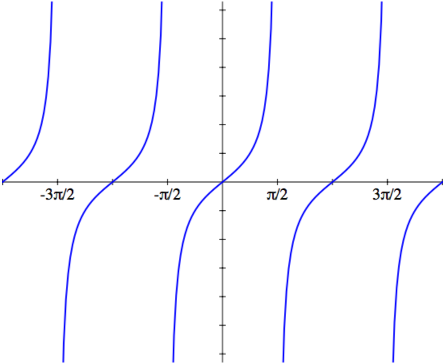
In videos 4.6 and 4.7 you saw a detailed treatment of the arcsin function. Specifically:

- How it is defined.
- What its domain and range are.
- How to compute $\sin(\arcsin(x))$ and $\arcsin(\sin(x))$ for different values of x.
- How to derive a formula for its derivative.

Today, we're going to do all of these things but for the arctan function.

The arctan function

Here's (part of) the graph of the tangent function.

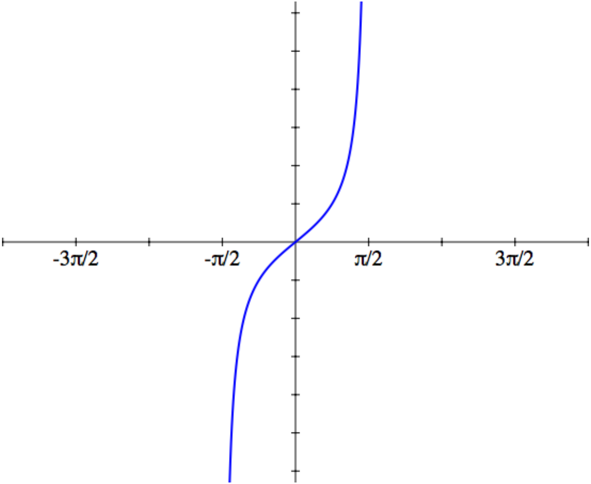


Question. Does this function have an inverse?

Problem. Find the largest interval containing 0 such that the restriction of tan to it is injective.

The arctan function

We define arctan to be the inverse of the function with this graph:



Question: What is the domain and range of arctan?

The arctan function

In symbols, that means we define the function arctan as the inverse of the function

$$g(x) = \tan x, \text{ restricted to the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In other words, if $x, y \in \mathbb{R}$, then

$$\arctan(y) = x \quad \iff \quad \begin{cases} ??? \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ ??? \end{cases}$$

Problem 1. What should be where the question marks are?

Problem 2. Sketch the graph of arctan.

The arctan function

To remind you:

$$\arctan(y) = x \quad \iff \quad \begin{cases} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan x = y \end{cases}$$

Problem 1: What is the function $\arctan\left(\tan\left(x\right)\right)$. What is it's domain and range? Graph it.

Problem 2: What is the function $\tan\left(\arctan\left(y\right)\right)$. What is it's domain and range? Graph it.

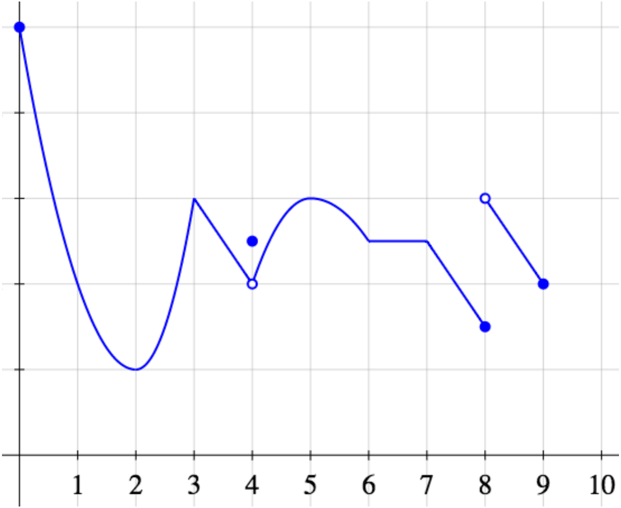
Derivative of arctan

Obtain (and prove) a formula for the derivative of arctan.

Hint: Differentiate the identity

$$\forall t \in \dots \qquad \tan(\arctan(t)) = t$$

Find local max/min of this function on $(0, 9)$ and $(3, 5)$
Find the max/min of this function on $[0, 9]$ and $[3, 5]$



Common misconceptions

We know the following information about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its entire domain.
- h is differentiable on its entire domain, except at 0.
- $h'(x) = 0 \quad \iff \quad x = -1$ or 1.

Problem. What can you conclude about h?

1. h has a local maximum at $x = -1$, or 1.
2. h has a local maximum at $x = -1, 0$, or 1.
3. h has a local maximum at $x = -4, 1, 0, 1$, or 4.
4. None of the above.

What can you conclude?

We know the following information about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

Problem. What can you conclude about $f'(0)$? Prove your answer.

Hint: Sketch a graph of what f might look like. Looking at your graph, make a conjecture.

To prove it, imitate the proof of the Local EVT from Video 5.3.

Note: The question does not say whether f is differentiable at 0, or anywhere else.

Let

$$g(x) = x^{2/3}(x-1)^3.$$

Problem 1: Find local maximums and minimums of g on $(-1, 2)$.

Problem 2: Find the maximum and minimum of g on $[-1, 2]$