MAT137 - Power series, Taylor Polynomials

- Today's lecture will assume you have watched videos 14.3,14.4

  For Monday's lecture, watch videos 14.5, 14.6, 14.7, 14.8

Warm up True and False

Let a_n be a sequence.

1. The power series $\sum_{n=0}^{\infty} a_n x^n$ can either only converge at $x = 0$, or only converge on an interval $(-R, R)$, or converge for all $x \in \mathbb{R}$.

2. Suppose the power series converge on $(-R, R]$, then $\frac{d}{dx} \sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $x \in (-R, R)$.

3. Suppose the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R, then the same holds for the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$. (Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exist).

What can you conclude?

Consider the power series $\sum_n a_n x^n$. Do not assume $a_n \geq 0$.

In each case, may the given series be absolutely convergent (AC)? conditionally convergent (CC)? divergent (D)? all of them?

IF	$\sum_n a_n 3^n$ is ...	AC	CC	D
THEN	$\sum_n a_n 2^n$ may be ...			
	$\sum_n a_n (-3)^n$ may be ...			
	$\sum_n a_n 4^n$ may be ...			

Challenge

We want to calculate the value of $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

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1. What is the value of the sum $\sum_{n=0}^{\infty} x^n$, when $|x| < 1$?
2. What is the relation between $\sum_{n}^{\infty} x^n$ and $\sum_{n}^{\infty} n\, x^{n-1}$?
3. Compute the value of the sum $\sum_{n=1}^{\infty} n\, x^{n-1}$.
4. Compute the value of the sum $\sum_{n=1}^{\infty} n\, x^{n}$.
5. Compute the value of the original series.

The definitions of Taylor polynomial: Warm up True and False

Let f be a function defined at and near $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.
Let P_n be the n-th Taylor polynomial for f at a.

Which ones of these is true?

1. P_n is an approximation for f of order n near a.
2. f is an approximation for P_n of order n near a.
3. $\lim_{x \to a} [f(x) - P_n(x)] = 0$
4. $\lim_{x \to a} \dfrac{f(x) - P_n(x)}{(x-a)^n} = 0$
5. $\exists$ a function R_n s.t. $f(x) = P_n(x) + R_n(x)$ and $\lim_{x \to a} \dfrac{R_n(x)}{(x-a)^n} = 0$.
6. $f^{(n)}(a) = P_n^{(n)}(a)$
7. $\forall k = 0, 1, 2, \ldots, n, \quad f^{(k)}(a) = P_n^{(k)}(a)$
8. If x is close to a, then $f(x) = P_n(x)$.

An explicit equation for Taylor polynomials

1. Find one polynomial P of degree 3 that satisfies

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Let f be a C^3 function. Find an explicit formula for the 3-rd Taylor polynomial for a function f at 0.

4. Let f be a C^∞ function, and let $n > 0$. Find an explicit formula for the n-th Taylor polynomial for a function f at 0.

Examples

Use your formula for Taylor polynomials (or think of a clever way) to obtain the degree n Taylor polynomial of the following functions at $x = 0$:

- $f(x) = \arctan(x)$
- $g(x) = \ln(1 - x)$
- $h(x) = e^x$
- $A(x) = \sin x$
- $B(x) = \cos x$