MAT137 - Alternating series, Conditional and absolute convergence

- Today's lecture will assume you have watched videos 13.18, 13.19, 14.1, 14.2

  For Tuesday's lecture, watch videos 14.3,14.4

Positive and negative terms – part 1

- Let $\sum a_n$ be a series.
- Call $\sum$ P.T. the sum of only the positive terms of the same series.
- Call $\sum$ N.T. the sum of only the negative terms of the same series.

IF $\sum$ P.T. is...	AND $\sum$ N.T. is...	THEN $\sum a_n$ may be...
CONV	CONV	
∞	CONV	
CONV	$-\infty$	
∞	$-\infty$	

Positive and negative terms – part 2

- Let $\sum a_n$ be a series.
- Call $\sum$ P.T. the sum of only the positive terms of the same series.
- Call $\sum$ N.T. the sum of only the negative terms of the same series.

	$\sum$ P.T.	$\sum$ N.T.
If $\sum a_n$ is CONV		
If $\sum \lvert a_n \rvert$ is CONV		
If $\sum a_n$ is ABS CONV		
If $\sum a_n$ is COND CONV		
If $\sum a_n = \infty$		
If $\sum a_n$ is DIV (oscillating)		

The Ratio Test

Theorem (Ratio Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with non-zero terms.

Suppose also that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ *exists, and equals a real number L.*

Then:

1. *IF $0 \leq L < 1$, THEN the series $\sum a_n$ converges absolutely.*
2. *IF $L > 1$, THEN the series $\sum a_n$ diverges.*
3. *IF $L = 1$, THEN we can't conclude anything about $\sum a_n$.*

Note: Note that $L = 0$ is fine here, in contrast to the LCT. Don't mix them up!

Ratio test – Convergent or divergent?

For the following series, try to use the ratio test to determine whether they converge.

If the ratio test is inconclusive, try another test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2\, 3^{n+1}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{n^n}$
5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Proof for the Ratio Test

To prove the Ratio Test, the idea is to "compare" a series $\sum a_n$ with a geometric series.

For any geometric series $\sum b_n$, where $b_n = r^n$ for some r, we have that

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|r|^{n+1}}{|r|^n} = |r|,$$

and that $\sum b_n$ converges if and only if $|r| < 1$. We will use this idea to prove the Ratio Test.

Problem 1: Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. Show that a_n for large n can be sandwiched by 2 geometric sequence:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t.}$$

$$\forall n > N, \qquad (L-\varepsilon)^{n-N}|a_N| < |a_n| < (L+\varepsilon)^{n-N}|a_N|$$

Problem 2: Now conclude the proof for the ratio test.

A New Test

We can use the same idea to create a new test. There is more than one way to "compare" a series $\sum a_n$ with a geometric series.

For any geometric series $\sum b_n$, where $b_n = r^n$ for some r, we have that

$$\text{For every } n \in \mathbb{N}, \quad \sqrt[n]{|b_n|} = \sqrt[n]{|r|^n} = |r|.$$

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$.

What can you say about $\sum a_n$ when L is greater than, less than, or equal to 1?

The Root Test

The Root Test

Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

Suppose also that $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, and equals a real number L.

Then:

1. IF $0 \leq L < 1$, THEN the series $\sum a_n$ converges absolutely.
2. IF $L > 1$, THEN the series $\sum a_n$ diverges.
3. IF $L = 1$, THEN we can't conclude anything about $\sum a_n$.

Replicate the proof for the Ratio Test to prove this.
Hint: first prove that a_n for big n can be sandwiched by 2 geometric sequences:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t.}$$

$$\forall n > N, \qquad (L-\varepsilon)^n < |a_n| < (L+\varepsilon)^n$$

Find the interval of convergence (i.e., not just the *radius* of convergence) of each of the following power series:

1. $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$
2. $$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$$
3. $$\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$$

What can you conclude?

Consider the power series $\sum_n a_n x^n$. Do not assume $a_n \geq 0$.

In each case, may the given series be absolutely convergent (AC)? conditionally convergent (CC)? divergent (D)? all of them?

IF	$\sum_n a_n 3^n$ is ...	AC	CC	D
THEN	$\sum_n a_n 2^n$ may be ...			
	$\sum_n a_n (-3)^n$ may be ...			
	$\sum_n a_n 4^n$ may be ...			

Writing functions as power series

Using the geometric series, we know how to write the function $F(x) = \dfrac{1}{1-x}$ as a power series centered at 0:

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Write the following functions as power series centered at 0:

1. $f(x) = \dfrac{1}{1+x}$
2. $g(x) = \dfrac{x}{1-x^2}$
3. $h(x) = \dfrac{1}{2-x}$
4. $G(x) = \ln(1+x)$

Challenge

We want to calculate the value of $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

---

1. What is the value of the sum $\sum_{n=0}^{\infty} x^n$, when $|x| < 1$?
2. What is the relation between $\sum_{n}^{\infty} x^n$ and $\sum_{n}^{\infty} n\, x^{n-1}$?
3. Compute the value of the sum $\sum_{n=1}^{\infty} n\, x^{n-1}$.
4. Compute the value of the sum $\sum_{n=1}^{\infty} n\, x^n$.
5. Compute the value of the original series.