- **Reminder:** Problem Set A is on the website now. It contains material that is not covered by Problem Sets 1 and 2, but that is covered by Test 1. It is not to be submitted, but it is very good practise. Do these problems before Test 1.

- Today's lecture will assume you have watched videos 2.21 - 2.22. We are done Playlist 2. Yaay!

  For Monday's lecture, watch videos 3.1-3.5, 3.8

How to study for the test

Where to practice:

1. Problem Set A
2. Lectures (for ALL instructors)
3. Playlist Practice Problems
4. Problem Sets
5. Problems from the book
   (If there is a specific topic that you are still struggling with, then hunt for more problems relating to that topic in the book)

To review the material, you have both the videos and the book.
If there is a concept you still don't understand from the videos, then go to the book for a different and (maybe) more detailed explanation.

Using the IVT and the EVT

Problem 1. Prove that the equation $x^3 - e^x - \cos(12x) + 100 = 0$ has at least two solutions

Problem 2: Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Prove that there exists at least one fixed point.
That means: $\exists x \in [0, 1]$ s.t. $f(x) = x$.

Using the IVT and the EVT

Problem 6. Let f be the function defined by

$$f(x) = \frac{(e^x + 10)\sin x}{x} + \cos x + 7$$

Does f have a maximum on each of the following intervals?

1. $[7, 20]$
2. $[0, 1]$
3. $(0, 1]$

For (3), we defined the function g to be the continuous extension of f on $[0, 1]$. So $g(x) = f(x)$ if $x \in (0, 1]$ and $g(0) = \lim_{x \to 0} f(x) = 19$. You should see that g(x) is continuous on $[0, 1]$. So now we can apply the EVT to conclude that g has a maximum on $[0, 1]$. We cannot yet say anything about whether f has a maximum on $(0, 1]$ or not because the maximum of g might only happen at $x = 0$. If so, then f has no maximum on $(0, 1]$. However, it happens that $f(0.25) = 19.136 > 19$, and so the maximum of g doesn't happen at $x = 0$. And so the maximum of g on $[0, 1]$ is the maximum of g on $(0, 1]$ which is the maximum of f on $(0, 1]$. And so f does have a maximum on $(0, 1]$.

Extrema

In each of the following cases, does the function f have a maximum and a minimum on the interval I?

1. $f(x) = x^2, \quad I = (-1, 1).$
2. $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \quad I = [2, 6]$
3. $f(x) = \dfrac{2x^2 - \sqrt{5x^7 + x^4 + 2}}{x^4 + 2x^2}, \quad I = [1, \infty)$

Solution for (3) is in the next slide

Solution for (3):

We can show that f has a minimum on $[1, \infty)$. First, notice that $f(1) < 0$.

Second, notice that $\lim_{x \to \infty} f(x) = 0$. Vaguely speaking, that is because when x is big, $5x^7 + x^4 + 2 \approx 5x^7$ and so the numerator behaves like $-\sqrt{5}x^{3.5}$ when x is big. The denominator behaves like x^4 when x is big and so $f(x)$ behaves like $\frac{-\sqrt{5}x^{3.5}}{x^4}$, which goes to 0 as x goes to ∞. To rigorously do this, divide the numerator and denominator by the highest power (x^4) and use limit laws.

Let $\epsilon = \frac{|f(1)|}{2}$, which is a positive number. Then you know there exists $N > 0$ such that: if $x > N$, then $|f(x)| < \frac{|f(1)|}{2}$. We can assume that $N > 1$ (why?). So in particular this means that $\forall x \in (N, \infty), f(x) > \frac{f(1)}{2}$.

Also, since f is continuous on $[1, N]$, we know by EVT that f has a minimum on $[1, N]$. So $\exists c \in [1, N]$ s.t. $\forall x \in [1, N], f(x) \geq f(c)$. In particular, $f(1) \geq f(c)$.

So:

- If $x \in [1, N]$, then $f(x) \geq f(c)$ by the above paragraph.
- If $x \in (N, \infty)$, $f(x) > \frac{f(1)}{2} > f(1) \geq f(c)$, and so $f(x) \geq f(c)$.

Hence, we finally get that $\forall x \in [1, \infty), f(x) \geq f(c)$, implying that f has a minimum on $[1, \infty)$

Can we do the same thing to show that f has a maximum on $[1, \infty)$? What goes wrong?
In fact, f is negative on $[1, \infty)$ and approaches 0 as x goes to ∞; so certainly, f does not have a maximum on $[1, \infty)$. (Prove that!)