MAT137 - Riemann sums, antiderivatives, and indefinite integrals

- Today's lecture will assume you have watched videos 7.5-7.12

  For Monday's lecture, watch videos 8.3, 8.4, 8.5, 8.6, 8.7

Important Theorems

Do these before next lecture.

Problem 1: Prove this theorem:

Theorem

Let f and g be integrable functions on $[a, b]$ satisfying $f \geq g$. Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Problem 2: Prove this theorem:

Theorem

Let f be a continuous on $[a, b]$ satisfying $f \geq 0$. Suppose also that $f(c) > 0$ for some $c \in (a, b)$. Then

$$\int_a^b f(x)dx > 0$$

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \qquad \int_0^4 f(x)dx = 9, \qquad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(t)dx$
3. $\int_2^0 f(x)dx$
4. $\int_2^4 f(x)dx$
5. $\int_{-2}^0 f(x)dx$
6. $\int_0^4 [f(x) - 2g(x)] \ dx$

Definite vs. indefinite integrals

Let f be a continuous function defined on some interval $[a, b]$.

Question 1: The notation $\int f(x)\,dx$ represents...?

1. A number.
2. A function.
3. A collection of functions.
4. None of the above.

Question 2: What do these notations represent (same options)...?

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_a^x f(t)\,dt$$

Using Riemann sums

Theorem

Let f be an integrable function defined on an interval $[a, b]$.

Let $P_1, P_2, P_3, \dots$ be a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \|P_n\| = 0.$$

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S^*_{P_n}(f).$$

Note that we don't need to specify which x_i^*'s we are using in each sub-interval of each partition. The result is true no matter which points we choose.

Riemann sums example

Calculate $\int_0^1 x^2 dx$ using Riemann sums.

Hints: Imitate the calculation in Video 7.11.

1. Let $f(x) = x^2$ on $[0, 1]$.
   Let $P_n = \{\text{breaking the interval into } n \text{ equal pieces}\}$.
2. What is Δx_i?
3. Write $S^*_{P_n}(f)$ as a sum when we choose x_i^* as the right end-point.
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.

Helpful formulas: $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$, $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

Initial Value Problem

Find a function f such that

- For every $x \in \mathbb{R}$, $f''(x) = \sin x + x^2$,
- $f'(0) = 5$,
- $f(0) = 7$.