MAT137 - Integrable functions

- Today's lecture will assume you have watched videos 7.5-7.12

  For Tuesday's lecture, watch videos 8.1, 8.2

Properties of lower and upper sums

Let f be a bounded function on $[a, b]$.

Problem 1: Let P be a partition. Prove that $L_P(f) \leq U_P(f)$

Problem 2: Let $P = \{a, b\}$ and $Q = \{a, c, b\}$ where $c \in (a, b)$. Prove that

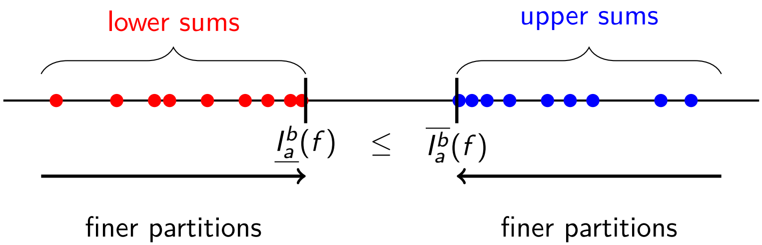
$$L_P(f) \leq L_Q(f), \quad U_P(f) \geq U_Q(f)$$

(This is true in general whenever $P \subseteq Q$).

Provlem 3: Prove that

$$\underline{I_a^b}(f) := \sup_P L_P(f) \leq \inf_P U_P(f) =: \overline{I_a^b}(f)$$

A summary of the properties of lower and upper sums



We say f is integrable on $[a, b]$ if

$$\underline{I_a^b}(f) = \overline{I_a^b}(f)$$

If so, we define the integral to be

$$\int_a^b f(x)dx = \underline{I_a^b}(f) = \overline{I_a^b}(f)$$

True or False

If False, fix it and prove the corrected version. If True, prove it

1. Let f and g be bounded functions on $[a, b]$. Then
$$\sup_{x\in[a,b]} [f(x) + g(x)] = \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x)$$
2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then
$$\sup_{x\in[a,c]} f(x) = \sup_{x\in[a,b]} f(x) + \sup_{x\in[b,c]} f(x)$$
3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:
$$\sup_{x\in[a,b]} (cf(x)) = c\left(\sup_{x\in[a,b]} f(x)\right)$$

Lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be some partition of $[a, b]$.

Draw yourself a picture of such a function and partition.

Which of the following expressions equal $L_P(f)$? What about $U_f(P)$?
(There may be more than one answer for each.)

1. $\displaystyle\sum_{i=0}^{N} \Delta x_i \, f(x_i)$
2. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_i)$
3. $\displaystyle\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$
4. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_{i+1})$
5. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$
6. $\displaystyle\sum_{i=0}^{N-1} \Delta x_{i+1} \, f(x_i)$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Constant functions

Do this as an exercise

Let f be the constant function C, defined on $[a, b]$.

Clearly, by looking at a picture, we see that the area under the graph of f is $C(b - a)$. Now we want to prove from the definition that f is indeed integrable and the integral is $C(b - a)$.

Problem 1: Fix an *arbitrary* partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[a, b]$, and *explicitly* compute $U_P(f)$ and $L_P(f)$.

Problem 2: Conclude that f is integrable on $[a, b]$ by showing that $\underline{I_a^b}(f) = \overline{I_a^b}(f) = C(b - a)$ and so

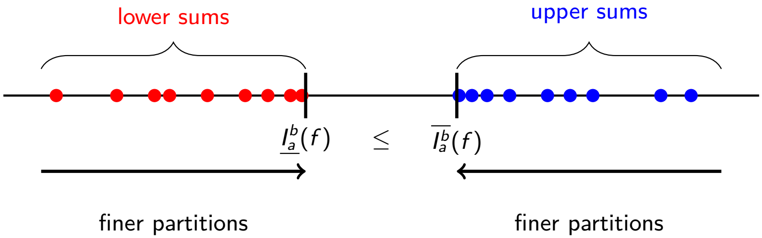
$$\int_a^b C\,dx = C(b - 1).$$

The "ε-characterization" of integrability

True or False?

Let f be a bounded function on $[a, b]$.

1. IF f is integrable on $[a, b]$
   THEN $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.
2. IF $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$,
   THEN f is integrable on $[a, b]$



Example 2: non-continuous

Do this as an exercise

Let $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{x_0, x_1, \ldots, x_N\}$ be any partition of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
2. Find a partition P such that $L_P(f) = 4.99$.
3. For any $\varepsilon > 0$, find a partition P such that $L_P(f) = 5 - \varepsilon$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

Important Theorems

Problem 1: Prove this theorem:

Theorem

Let f and g be integrable functions on $[a, b]$ satisfying $f \geq g$. Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Problem 2: Prove this theorem:

Theorem

Let f be a continuous on $[a, b]$ satisfying $f \geq 0$. Suppose also hat $f(c) > 0$ for some $c \in (a, b)$. Then

$$\int_a^b f(x)dx > 0$$