MAT137 - Properties of Sequences, Theorems about sequences

- Today's lecture will assume you have watched videos 11.3, 11.4, 11.5, 11.6

  For Tuesday's lecture, watch videos 11.7, 11.8

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
11. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except finitely many.
12. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many of the terms of the sequence.
13. Every interval that contains L must contain all but finitely many of the terms of the sequence.
14. Every open interval that contains L must contain all but finitely many of the terms of the sequence.

True or False – Monotonic sequences vs. functions

Let f be a function defined at least on $[1, \infty)$.
We define a sequence by $a_n = f(n)$.

1. IF f is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.
2. IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN f is increasing.

(If you think one of them is true, try to prove it.
If you think one of them is false, give a counterexample.)

Continuous functions respect sequence convergence

The following is a very useful theorem. Write a proof for it!

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.

- IF $\begin{cases} \{a_n\}_{n=0}^{\infty} \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

1. Write the definitions of the two hypotheses and the conclusion.
2. Using the definition of the conclusion, figure out the structure of the proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Quick review – True or False?

1. If a sequence is convergent, then it is bounded.
2. If a sequence is convergent, then it is eventually monotonic.
3. If a sequence diverges and is increasing, then there exists $n \in \mathbb{N}$ such that $a_n > 100$.
4. If $\lim_{n\to\infty} a_n = L$, then $a_n < L + 1$ for all n.
5. If a sequence is bounded and eventually monotonic, then it converges.
6. If $\lim_{n\to\infty} a_{2n} = L$, then $\lim_{n\to\infty} a_n = L$.

A recursively-defined sequence

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \geq 1, & a_{n+1} = \dfrac{a_n + 2}{a_n + 3} \end{cases}$$

Compute a_1, a_2, and a_3.

Does it converge?

Is this proof correct?

Let $\{a_n\}_{n=0}^{\infty}$ be the sequence in the previous slide.

Claim:

$\{a_n\}_{n=0}^{\infty}$ converges to $-1+\sqrt{3}$.

Proof.

Let $L = \lim_{n\to\infty} a_n$.
Starting with the recurrence relation and taking limits of both sides, we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[\frac{a_n+2}{a_n+3}\right] \implies L = \frac{L+2}{L+3} \implies L^2+2L-2=0$$

Solving the quadratic yields $L = -1 \pm \sqrt{3}$.

Every term of the sequence is postive, so L cannot be negative. So we conclude that $L = -1+\sqrt{3}$. □

Another recursively-defined sequence

Consider the sequence b_n defined by

$$\begin{cases} & b_0 = 1 \\ \forall n \geq 1, & b_{n+1} = 1 - b_n \end{cases}$$

1. Using the same technique as in the previous slide, compute the limit of the sequence.
2. **AFTER** you have computed the limit, compute the first five terms of the sequence by hand.

What happened?

The first recursive sequence, done correctly.

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \geq 1, & a_{n+1} = \dfrac{a_n + 2}{a_n + 3} \end{cases}$$

1. Prove $\{a_n\}_{n=0}^{\infty}$ is bounded below by 0.
2. Prove $\{a_n\}_{n=0}^{\infty}$ is decreasing (use induction).
3. Prove $\{a_n\}_{n=0}^{\infty}$ is convergent (use a theorem).
4. Now the calculation in the earlier slide is correct and justified.

Another recursive seqeunce

Do this as an exercise.

Define the sequences a_n and b_n in the following way:

$$\begin{cases} & a_0 = 1, \;\; b_0 = 2 \\ \forall n \geq 1, & a_{n+1} = \sqrt{a_n b_n}, \;\; b_{n+1} = \frac{a_n + b_n}{2} \end{cases}$$

Show that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist and are equal.