MAT137 - Week 5 Lecture 1

- Today's lecture will assume you have watched videos 2.14 - 2.18
  For Tuesday's lecture, watch videos 2.19 - 2.20.

Quick questions. Did you watch the videos?

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

True or False

- If $\lim_{x \to a} f(x)$ exist, then there exists a unique value L to assign to $f(a)$ such that f is continuous at a.

Let f and g be two functions that are equal at an interval centered at a except at a.

- $\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a} g(x)$ exist, and if so, they are equal.
- f is continuous at a iff g is continuous at a.

- f is continuous at a iff $|f|$ is continuous at a.

Absolute value function

We know that $g(x) = |x|$ is continuous everywhere on $\mathbb{R}$. (Prove it!)

Is this true or false? Prove or Disprove.

Claim

Let f be any function defined on $\mathbb{R}$. Suppose $\lim_{x \to a} f(x)$ exists. Then $\lim_{x \to a} |f(x)|$ exists and

$$\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)|$$

Absolute value function

Theorem

Let f be any function defined on $\mathbb{R}$. Suppose $\lim_{x \to a} f(x)$ exists. Then $\lim_{x \to a} |f(x)|$ exists and

$$\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)|$$

Corollary

If f is continuous at a, then $|f|$ is continuous at a.

So we have seen that $\lim_{x \to a} f(x)$ exists implies $\lim_{x \to a} |f(x)|$ exists, but the other direction fails. Does the backward direction always fail?

What if $\lim_{x \to a} |f(x)|$ exists and equals 0, then what can we say about $\lim_{x \to a} f(x)$?

Use this to prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

New continuous functions from old ones

You already know that sums, products, composititions, and certain quotients of continuous functions are continuous. Now, you will prove another similar result.

Theorem

Let f and g be continuous functions.

THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You may use anything you've learned thus far to prove this.

Hint: There is a way to prove this quickly, and without any epsilons or deltas.

A new theorem about limits

In one of the videos you learned about the Squeeze Theorem. Now we're going to prove a similar (and simpler) theorem.

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined at least on an interval centred at a, except possibly at a.

If

- *for x close to a (but not a), $f(x) \geq g(x)$,*
- $\lim_{x \to a} g(x) = \infty$,

then $\lim_{x \to a} f(x) = \infty$

A new theorem about limits

When working on this, do the following things, in this order.

1. Replace the hypothesis in the first bullet point with a more precise mathematical statement (i.e., there should be a quantifier).
2. Write down the precise definition of the second hypothesis.
3. Write down the precise definition of what you have to prove.
4. Write down the structure that your proof must have.
5. Start thinking about the problem (i.e., do some rough work to figure out how to prove it).
6. Write the complete proof.