MAT137 - Integrable functions

- **Reminder:** Problem Set 5 is due this Thursday at 11:59pm.

- Today's lecture will assume you have watched videos 7.5, 7.6, 7.7, 7.8, 7.9.

  For Monday's lecture, watch videos 7.5, 7.6, 7.7, 7.8, 7.9 again! and also 7.10, 7.11, 7.12

Useful equivalent definition of supremum

Recall the other equivalent definition of supremum:

Let M be an upper bound for the set A.
Then $M = \sup A$ if and only if the following is satisfied:

$$\forall L < M, \exists x \in A \text{ such that } L < x \leq M$$

Prove this lemma.

Lemma

Let M be an upper bound for the set A.
Then $M = \sup A$ if and only if the following is satisfied:

$$\forall \epsilon > 0, \exists x \in A \text{ such that } M - \epsilon < x \leq M$$

Useful Lemma

Problem: Is this true or false?

Let f be a bounded function on $[a, b]$.
Let $M \in \mathbb{R}$ satisfy $\forall x \in [a, b], \;\; f(x) < M$.
Then $\sup\limits_{x \in [a,b]} f(x) < M$

FALSE. Prove the corrected version:

Lemma

Let f be a bounded function on $[a, b]$.
Let $M \in \mathbb{R}$ satisfy $\forall x \in [a, b], \;\; f(x) < M$.
Then $\sup\limits_{x \in [a,b]} f(x) \leq M$

True or False

We will do this slide next lecture. Please attempt it before.

If False, fix it and prove the corrected version. If True, prove it

1. Let f and g be bounded functions on $[a, b]$. Then

$$\sup_{x\in[a,b]} [f(x) + g(x)] = \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x)$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\sup_{x\in[a,c]} f(x) = \sup_{x\in[a,b]} f(x) + \sup_{x\in[b,c]} f(x)$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\sup_{x\in[a,b]} (cf(x)) = c \left(\sup_{x\in[a,b]} f(x) \right)$$

Warm up: partitions

Which of the following are partitions of $[0,2]$?

1. $[0,2]$
2. $(0,2)$
3. $\{0,2\}$
4. $\{1,2\}$
5. $\{0,1.5,1.6,1.7,1.8,1.9,2\}$

Warm up: lower and upper sums

Let $f(x) = \cos x$.

Consider the partition $P = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ of the interval $[0, 2\pi]$.

Problem. Compute $L_P(f)$ and $U_P(f)$.

Properties of lower and upper sums

Let f be a bounded function on $[a, b]$.

Problem 1: Let P be a partition. Prove that $L_P(f) \leq U_P(f)$

Problem 2: Let $P = \{a, b\}$ and $Q = \{a, c, b\}$ where $c \in (a, b)$. Prove that

$$L_P(f) \leq L_Q(f), \quad U_P(f) \geq U_Q(f)$$

(This is true in general whenever $P \subseteq Q$).

Do this as an exercise. We will do it next lecture.

Provlem 3: Prove that

$$\underline{I_a^b}(f) := \sup_P L_P(f) \leq \inf_P U_P(f) =: \overline{I_a^b}(f)$$