MAT137 - Sums and sigmas, Suprema and infima

- My office hours has changed. Now they are Mon 2-4 in PG 003.
- **Reminder:** Problem Set 5 is due this Thursday at 11:59pm.
- We will finally start integration! Forget EVERYTHING you "know" about integration.
- Today's lecture will assume you have watched videos 7.1-7.4.

  For Tuesday's lecture, watch videos 7.5, 7.6, 7.7, 7.8, 7.9. They are *hard*.

Warm-up: Sigma Notation

Recall:

$$\sum_{i=1}^{n} a_i \;=\; a_1 + a_2 + a_3 + \ldots + a_n$$

Compute

$$\sum_{i=2}^{4}(2i+1)$$

Sigma notation exercise

Consider the following sum written in sigma notation

$$\sum_{j=0}^{N} \frac{x^j}{2j+1}.$$

Does the value of this expression depend on...

1. ...x only?
2. ...N only?
3. ...j only?
4. ...x and j?
5. ...j and N?
6. ...x and N?

This defines a function of x:

$$f(x) := \sum_{j=0}^{N} \frac{x^j}{2j+1}.$$

Sigma notation exercise

Do this slide as an exercise.

Write the following sums as a single sum in sigma notation. There may be many ways to do each of them.

1. $2^7 + 3^7 + 4^7 + 5^7 + 6^7 + 7^7$
2. $3 + 5 + 7 + 9 + \cdots 75 + 77$
3. $\displaystyle\sum_{i=1}^{100} a_i - \sum_{i=1}^{77} a_i$
4. $\cos(0) - \cos(2) + \cos(4) - \cos(6) + \cos(8) - \cdots \pm \cos(2N)$
5. $\displaystyle -\frac{2x^4}{3!} + \frac{3x^5}{4!} - \frac{4x^6}{5!} + \cdots - \frac{98x^{100}}{99!}$

Double sums

Compute:

1. $\sum_{i=1}^{N} \sum_{k=1}^{N} 1$
2. $\sum_{i=1}^{N} \sum_{k=1}^{i} 1$
3. $\sum_{i=1}^{N} \sum_{k=1}^{i} i$
4. $\sum_{i=1}^{N} \sum_{k=1}^{i} k$
5. $\sum_{i=1}^{N} \sum_{k=1}^{i} (ik)$

The following formulas may be useful:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}, \quad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4}$$

Can we change the order of two sums?

- $A_{i,k}$ is a function of 2 variables.
  For example, $A_{i,k} = \frac{i}{k+i^2}$.
- Decide what to write instead of each "?" so that the following identity is true:

1. $$\sum_{i=1}^{N} \sum_{k=1}^{M} A_{i,k} = \sum_{k=?}^{?} \sum_{i=?}^{?} A_{i,k}$$
2. $$\sum_{i=1}^{N} \sum_{k=1}^{i} A_{i,k} = \sum_{k=?}^{?} \sum_{i=?}^{?} A_{i,k}$$

Let $A \subset \mathbb{R}$ that is bounded from above and nonempty.

Let $U := \{b \in \mathbb{R} : \forall a \in A, b \geq a\}$

Problem 1: Define $\sup A$ and $\inf U$ and explain why they both exist.

Problem 2: Show that U has a minimum and conclude that

$$\inf U = \sup A$$

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Infima and suprema exercise

For each of the following sets of real numbers

- find its supremum or convince yourself it does not exist;
- do the same for the infimum;
- find the maximum and minimum, if they exist.

1. $A_1 = (0, 7]$
2. $A_2 = \{7, 8, 9\}$
3. $A_3 = \{\, x \in \mathbb{R} \,:\, x < 0 \text{ or } x \geq 7 \,\}$
4. $A_4 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \left\{ \frac{1}{n} \,:\, n \in \mathbb{N} \right\}$
5. $A_5 = \{\dots, \frac{1}{343}, \frac{1}{49}, \frac{1}{7}, 1, 7, 49, 343, \dots\} = \{\, 7^n \,:\, n \in \mathbb{Z} \,\}.$

Empty set questions

1. Does $\emptyset$ have an upper bound ?
2. Does $\emptyset$ have a supremum?
3. Does $\emptyset$ have a maximum?
4. Is $\emptyset$ bounded above?

Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- a is an **upper bound** of A means: $\forall x \in A, x \leq a$.
- a is the **least upper bound** (lub) or **supremum** (sup) of A means
  - a is an upper bound of A, and
  - there are no smaller upper bounds.

Infima and suprema exercise

Recall again that M is the supremum of a set A if...

1. ...M is an upper bound of A;
2. ...and there are no smaller upper bounds of A.
   (Write down what that means precisely)

With this in mind, **assume** M **is an upper bound for a set** A.
Which of the following is equivalent to "M is the supremum of A"?

1. If L is an upper bound of A, then $L \geq M$.
2. $\forall L \geq M, \quad L$ is an upper bound of A.
3. $\forall L \leq M, \exists x \in A$ such that $L < x$.
4. $\forall L < M, \exists x \in A$ such that $L < x$.
5. $\forall L < M, \exists x \in A$ such that $L < x \leq M$.
6. $\forall \epsilon > 0, \exists x \in A$ such that $M - \epsilon < x \leq M$.