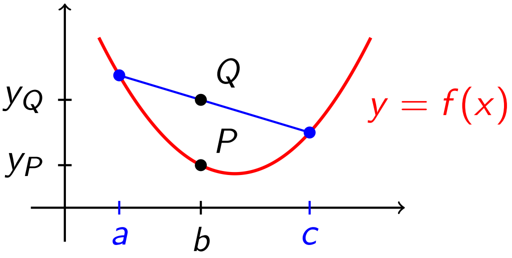
MAT137 - Week 12 Lecture 3

- Please fill out the feedback form by Friday December 13. Your responses are anonymous. The link is available in the announcement titled "Mid-year course feedback".
- Today's lecture will assume you have watched videos 6.13, 6.14, 6.15.

"Secant segments are above the graph"

Let f be a function defined on an interval I.
In Video 6.11 you learned that an alternative way to define "f is concave up on I" is to say that "the secant segments stay above the graph".



Rewrite this as a precise mathematical statement of the form

$$\text{"}\forall a, b, c \in I, \quad a < b < c \implies \boxed{\text{an inequality involving } f,\ a,\ b,\ c}\text{"}$$

Prove the forward direction!: If f is concave up on I, then it satisfies the above statement.

Monotonicity and concavity

Let $f(x) = xe^{-x^2/2}$.

1. Find the intervals where f is increasing or decreasing, and its local extrema.
2. Find the intervals where f is concave up or concave down, and its inflection points.
3. Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
4. Using this information, sketch the graph of f.

To save you time, here are the derivatives of f:

$$f'(x) = -e^{-x^2/2}(x^2 - 1) \qquad f''(x) = e^{-x^2/2}x(x^2 - 3)$$

Lots of asymptotes

Consider the function $f(x) = \dfrac{x-1}{\sqrt{4x^2-1}}$.

Here are its first two derivatives, fully factored:

$$f'(x) = \frac{4x-1}{(4x^2-1)^{3/2}} \qquad f''(x) = -\frac{4(8x^2-3x+1)}{(4x^2-1)^{5/2}}.$$

1. Determine the domain of f.
2. This function has *four* asymptotes. Find them!
3. Use f' to study its monotonocity.
4. Use f'' to study its concavity.
5. Sketch the graph of f.

Unusual examples

Construct a function f such that

- the domain of f is at least $(0, \infty)$
- f is continuous and concave up on its domain
- $\lim_{x \to \infty} f(x) = -\infty$

Construct a function g such that

- the domain of g is $\mathbb{R}$
- g is continuous
- g has a local minimum $x = 0$
- g has an inflection point at $x = 0$

Asymptotics

Definition

We say f is asymptotic to g at ∞ if

$$\lim_{x \to \infty} (f(x) - g(x)) = 0$$

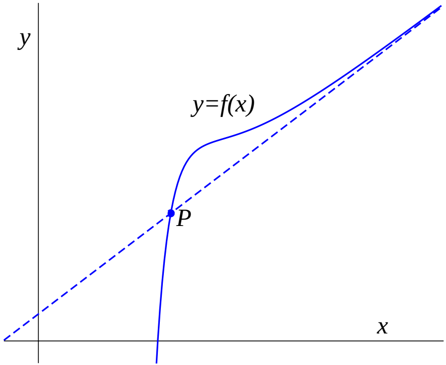
There's a similar definition for f asymptotic to g at $-\infty$.

We are interested in the case when f is asymptotic to a line L. If L is horizontal, then we say f has a horizontal asymptote.

Example of f asymptotic to a line L

$$f(x) = 3x + 4 + \frac{2x - 10}{x^2}$$

This is an example where f is asymptotic to a line L. Since L is not horizontal, we say f has a slant asymptote. Find L and P.



Problem 1: Does this function have a slant asymptote? (Is this function asymptotic to a line at ∞ or $-\infty$).

$$f(x) = \sqrt{x^2 + 5x}$$

Problem 2: Let's generalize this. Given a function f, how can we know if it has a slant asymptote? That is: does there exist numbers $a, b \in \mathbb{R}$ such that:

$$\lim_{x \to \pm\infty} [f(x) - (ax + b)] = 0$$

Unexpected asymptotes

Steps to take to find if a function f has a slant asymptote:

1. Does $\lim_{x\to\pm\infty} \frac{f(x)}{x}$ exist? If no, then no slant asymptote. If yes, call it a.
2. Does $\lim_{x\to\pm\infty} x\left(\frac{f(x)}{x} - a\right)$ exist? If no, then no slant asymptote. If yes, call it b and $L(x) = ax + b$ is a slant asymptote at $\pm\infty$.

Does this function have asymptotes? Find all asymptotes.

$$f(x) = x + \sqrt{x^2 + x}$$

A tricky function to graph

The function $f(x) = xe^{1/x}$ is weird. To save you time, here are its derivatives:

$$f'(x) = \frac{x-1}{x}e^{1/x} \qquad f''(x) = \frac{e^{1/x}}{x^3}$$

1. Examine the behaviour f as $x \to \pm\infty$.
   There is an asymptote, but it's tricky to see.
2. Carefully examine the behaviour of f as $x \to 0^+$ and $x \to 0^-$.
   They are very different.
3. Use f' to study its monotonocity.
4. Use f'' to study its concavity.
5. Sketch the graph of f.

Show that f is always above its slant asymptote for $x > 0$ and always below it for $x < 0$.