MAT137 - LCT and Series

- Today's lecture will assume you have watched videos 13.2, 13.3, 13.4, 13.5, 13.6, 13.7

  For Tuesday's lecture, watch videos 13.8, 13.9

Warm up for LCT

Are these improper integrals convergent or divergent? Can we use the BCT for both?

1. $$\int_1^{\infty} \frac{1}{x^2+1}dx$$
2. $$\int_2^{\infty} \frac{1}{x^2-1}dx$$

A variation on LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypotheses to $L = 0$?

1. Write down the new version of this theorem (different conclusion).
2. Prove it.

Hint: If $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, what is larger $f(x)$ or $g(x)$?

Another variation of LCT for type II Improper integrals

You have learned LCT for Type I improper integrals (Bounded functions on unbounded domains).
This is a variation of LCT for Type II improper integrals (Unbounded functions on bounded domains).

Theorem (Limit-Comparison Test)

Let f and g be positive, continuous functions on $(0,1]$.

- IF the limit $L = \lim_{x \to 0^+} \dfrac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ are both convergent or both divergent.

Prove this!

Convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$
3. $\displaystyle\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$
4. $\displaystyle\int_0^1 \cot x\,dx$
5. $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$
6. $\displaystyle\int_0^1 \frac{\arctan x}{x^{1.1}}\,dx$

Do the following series converge or diverge?

1. $\sum_{n=0}^{\infty} \sin(n\pi)$

2. $\sum_{n=0}^{\infty} \cos(n\pi)$

Telescoping series

We want to calculate the value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

1. Find a formula for the k-th partial sum $S_k = \sum_{n=1}^{k} \frac{1}{n^2+2n}$.

   Hint: First write $\frac{1}{n^2+2n} = \frac{A}{n} + \frac{B}{n+2}$.

2. Using the definition of a series, compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}.$$

For convergence, it doesn't matter where you start.

I leave this as an exercise for you

Something we know about sequences is that if you care about the limit of a sequence, it doesn't matter where you start.

In other words, $\{a_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=17}^{\infty}$ have the same limit (if it exists), and the fact that the first 16 terms of the latter sequence are "missing" doesn't matter.

The same is true of series, in the following sense.

Claim.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence, and $M > 1$ is an integer. Then:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=M}^{\infty} a_n \text{ converges.}$$

Problem. Prove this, using the definition of series convergence.

Proving a property about series from the definition

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $c \in \mathbb{R}$. Prove that

- IF $\sum_{n=0}^{\infty} a_n$ is convergent.
- THEN $\sum_{n=0}^{\infty} (ca_n)$ is also convergent and

$$\sum_{n=0}^{\infty} (ca_n) = c \left[\sum_{n=0}^{\infty} a_n \right].$$

Write a proof directly from the definition of series.

What is wrong with this calculation? Fix it.

Claim.

$$\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) = \ln 2$$

"Justification"

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) &= \sum_{n=2}^{\infty} [\ln(n) - \ln(n+1)] \\
&= \sum_{n=2}^{\infty} \ln(n) \; - \; \sum_{n=2}^{\infty} \ln(n+1) \\
&= (\ln 2 + \ln 3 + \ln 4 + \ldots) \; - \; (\ln 3 + \ln 4 + \ln 5 + \ldots) \\
&= \ln 2
\end{aligned}$$