

Instructor: B. Khesin

Course MAT461S
Spring 2026
“Hamiltonian Mechanics”

Problem Set 2 (due Tuesday Feb. 24, before the lecture),
17 points in total:

Main source: [Ar] textbook by V. Arnold, see

https://www.math.toronto.edu/khesin/biblio/arnold_Math_Methods89.pdf

1. Obtain the Euler-Lagrange equation for the brachistochrone problem (the curve minimizing the travel time between two fixed points on the plane with different x and y coordinates. Verify that cycloids are solutions (google the necessary equations and properties of cycloids).

Find the extremal(s) among functions $y(t)$ for each of the following:

2.

$$\Phi(y) = \int_1^2 (y' + t^2(y')^2) dt \quad \text{when } y(1) = 0, y(2) \text{ free}$$

3.

$$\Phi(y) = \int_0^1 \left(\frac{1}{2}(y')^2 + yy' + y + y' \right) dx \quad \text{when } y(0), y(1) \text{ are both free}$$

4 (p.62 of [Ar]). Let $f(x) = x^\alpha/\alpha$. Show that the Legendre transform of f is $g(p) = p^\beta/\beta$, where $(1/\alpha) + (1/\beta) = 1$. (Here $\alpha > 1$ and $\beta > 1$.)

5. Prove the following simplified version of the Poincaré recurrence / Birkhoff ergodic theorem: Let $f(\phi) = \sum_{|k| \leq K} a_k e^{ik\phi}$ be any trigonometric polynomial on the circle S^1 , and $\mathbf{f} := (1/2\pi) \int_{S^1} f(\phi) d\phi$ is its space average. Let α be an angle non-commensurable with π , i.e. $\alpha/\pi \notin \mathbf{Q}$. Define a new function \tilde{f} on the circle by

$$\tilde{f}(\phi) := \lim_{N \rightarrow \infty} \frac{f(\phi) + f(\phi + \alpha) + \dots + f(\phi + (N-1)\alpha)}{N}.$$

(This function is called the time average of f .) Then for all $\phi \in S^1$ this limit exists and $\tilde{f}(\phi) = \mathbf{f}$.

Hint: use the formula for the sum of a finite geometric progression.