Instructor: B. Khesin

Course MAT461S
Spring 2025
"Hamiltonian Mechanics"

Problem Set 2 (due Tuesday Feb. 25):

Only the best 5 problems count (4pts each).

Main source: [Ar] textbook by V. Arnold, see

`https://www.math.toronto.edu/khesin/biblio/arnold_Math_Methods89.pdf`

1. Obtain the Euler-Lagrange equation for the brachistochrone problem (the curve minimizing the travel time between two fixed points on the plane with different x and y coordinates. Verify that cycloids are solutions (google the necessary equations and properties of cycloids).

Find the extremal(s) among functions $y(t)$ for each of the following:

2.

$$\Phi(y) = \int_0^1 \frac{\sqrt{1+(y')^2}}{y}\, dt \qquad \text{with} \;\; y(0)=0,\; y(1)=\sqrt{3}$$

(Hint: use the substitution $y' = \tan z$.)

3.

$$\Phi(y) = \int_1^2 (y' + t^2 (y')^2)\, dt \qquad \text{when} \;\; y(1)=0,\; y(2) \text{ free}$$

4.

$$\Phi(y) = \int_0^1 (\frac{1}{2}(y')^2 + yy' + y + y')\, dx \qquad \text{when} \;\; y(0),\; y(1) \text{ are both free}$$

5 (p.62 of [Ar]). Let $f(x) = x^\alpha/\alpha$. Show that the Legendre transform of f is $g(p) = p^\beta/\beta$, where $(1/\alpha)+(1/\beta)=1$. (Here $\alpha>1$ and $\beta>1$.)

6. Prove the following simplified version of the Poincaré recurrence / Birkhoff ergodic theorem: Let $f(\phi) = \sum_{|k|\le K} a_k e^{ik\phi}$ be any trigonometric polynomial on the circle S^1, and $\mathbf{f} := (1/2\pi)\int_{S^1} f(\phi)\, d\phi$ is its space average. Let α be an angle non-commensurable with π, i.e. $\alpha/\pi \notin \mathbf{Q}$. Define a new function $\tilde f$ on the circle by

$$\tilde f(\phi) := \lim_{N\to\infty} \frac{f(\phi) + f(\phi+\alpha) + \ldots + f(\phi + (N-1)\alpha)}{N}.$$

(This function is called the time average of f.) Then for all $\phi \in S^1$ this limit exists and $\tilde f(\phi) = \mathbf{f}$.

Hint: use the formula for the sum of a finite geometric progression.