Instructor: B. Khesin

Course MAT461S
Spring 2025
"Hamiltonian Mechanics"

Problem Set 1 (due Tuesday Feb. 4):

Only the best 5 problems count (4pts each).

Main source: [Ar] textbook by V. Arnold, see

`https://www.math.toronto.edu/khesin/biblio/arnold_Math_Methods89.pdf`

1. Sketch the phase portrait of the Newton equation with potential

$$U(x) = x^2 \sin(1/x)\,.$$

2 (p.20 of [Ar]). For a Newton's equation let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period of motion along this curve is equal to $T = dS/dE$.

3. Prove the theorem: Trajectories for the central force proportional to $|w|^a$ for $w \in \mathbf{C}$ are taken by the map $Z = w^\alpha$ with $\alpha = (a+3)/2$ into trajectories with the force proportional to $|Z|^A$. Find the relation of a and A.

4 (p.51 of [Ar]). Let $U(r)$ be a homogeneous function of degree ν. For such a potential if γ is an orbit, then $\alpha\gamma$ is an orbit, $\alpha \in \mathbf{R}_+$. Find the ratio of circulation times for γ and $\alpha\gamma$. Derive from this the isochronicity of the pendulum oscillations (Hooke's law, $\nu = 2$) and Kepler's third law ($\nu = -1$).

5. Derive an approximate value of the third cosmic velocity (the escape velocity from our solar system from the Earth orbit, regarded as a circle centered at the Sun). Find the required data (distances, masses, etc) by googling on the internet. (There are several possible definitions of the third velocity, so your answers might be different depending on the definition used.)

6. Describe the main geometric property of parabola (the set of points equidistant from a fixed point and a fixed line) as a conic section via Dandelin spheres.