The Legendre-Fenchel Transform and Involution

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1 Introduction

The Legendre-Fenchel transform is a fundamental tool in convex analysis, physics, and optimization. A primary question regarding this transform is whether it acts as an involution—that is, whether applying the transform twice returns the original function ($f^{**} = f$).

2 Formal Definition

For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the Legendre-Fenchel transform f^* is defined as:

$$f^*(p) = \sup_{x \in \mathbb{R}^n} \{\langle p, x \rangle - f(x)\} \tag{1}$$

By defining $f(x) = \infty$ for points outside the function's natural domain, we treat the function as being defined over the entire space $\mathbb{R}^n$.

3 The Requirement for Involution

The transform is an involution ($f^{**} = f$) if and only if the function f belongs to the class of **proper, lower semi-continuous (LSC), convex functions**.

3.1 The Role of Infinity

Setting $f(x) = \infty$ outside the domain handles restricted domains gracefully. In the supremum calculation, points where $f(x) = \infty$ yield $-\infty$ inside the curly braces, ensuring they never contribute to the maximum. This "infinity strategy" ensures that:

- The **domain** of f determines the **growth rate** of f^*.
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3.2 Lower Semi-Continuity (Closure)

Even if a function is convex, it must be LSC (also called "closed") for the involution to be perfect. If a function is not LSC, the biconjugate f^{**} will return the *closure* of the function, effectively "filling in" the boundary points.

4 Example: The Box Function

Consider the indicator-style function on a restricted domain $x \in [-1, 1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1, 1] \\ \infty & \text{otherwise} \end{cases} \tag{2}$$

Step 1: First Transform

$$f^*(p) = \sup_{x \in [-1,1]} \{px - 0\} = |p| \tag{3}$$

Step 2: Second Transform (Biconjugate)

$$f^{**}(x) = \sup_{p \in \mathbb{R}} \{xp - |p|\} \tag{4}$$

If $|x| \leq 1$, the supremum is 0. If $|x| > 1$, the expression can be made arbitrarily large by increasing p. Thus:

$$f^{**}(x) = \begin{cases} 0 & \text{if } x \in [-1, 1] \\ \infty & \text{otherwise} \end{cases} \tag{5}$$

Here, $f^{**} = f$, demonstrating a perfect involution.

5 Summary Table

Properties of f	**Resulting f^{**}**	**Involution?**
Convex, LSC, Proper	$f^{**} = f$	Yes
Convex, Proper, *not* LSC	$f^{**} = \text{cl}(f)$	No (Closure only)
Non-Convex	$f^{**} = \text{conv}(\text{cl}(f))$	No (Convex Hull)