Lecture 6

Thm 2 (Bohlin 1911) Assume $w \in \mathbb{C}$ moves according to Hooke $\ddot{w} = -w$. Let $z = w^2$ and new time τ is such that the corresp sectorial area has const speed. Then $z(\tau)$ satisfies Kepler: $\frac{d^2 z}{d\tau^2} = -c \frac{z}{|z|^3}$

Pf The law of areas: $\frac{|w|^2 d\varphi}{dt} = \text{const}$, $\frac{|z|^2 d\varphi}{d\tau} = \text{const}$

$$\frac{d\tau}{dt} = \frac{|z|^2}{|w|^2}. \text{ Then } \frac{d}{d\tau} = \frac{1}{|w|^2} \frac{d}{dt} \text{ and}$$

$$\frac{d^2 z}{d\tau^2} = \frac{1}{w\bar{w}} \frac{d}{dt}\left(\frac{1}{w\bar{w}} \frac{dw^2}{dt}\right) = \frac{2}{w\bar{w}} \frac{d}{dt}\left(\frac{1}{\bar{w}} \frac{dw}{dt}\right)$$

$$= -\frac{2}{w\bar{w}}\left(\frac{1}{\bar{w}^2}\frac{dw}{dt}\frac{d\bar{w}}{dt} + \frac{w}{\bar{w}}\right) = -2w^{-1}\bar{w}^{-3}\left(|\dot{w}|^2 + |w|^2\right)$$

used $\ddot{w} = -w$

Here $\boxed{|\dot{w}|^2 + |w|^2 = 2E}$ on the trajectory (conserv. of energy)

$$\underset{2E}{\overset{\shortparallel}{}} \qquad = -4E\, w^{-1}\bar{w}^{-3}$$

Since $\frac{z}{|z|^3} = \frac{w^2}{|w^2|^3} = \frac{w^2}{w^3\bar{w}^3} = w^{-1}\bar{w}^{-3}$, $c = 4E$

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Thm 3 Trajectories for force $\sim |w|^a$ are taken by $z = w^\alpha$, $\alpha = \frac{a+3}{2}$

into traj's w/ force $\sim |z|^A$.

(you'll find $A = f(a)$)

Similarity in physics

Observation Let $\bar{r}(t)$ satisfy $m \frac{d^2\bar{r}}{dt^2} = -\frac{\partial U}{\partial \bar{r}}$

take $t_1 = \alpha t$, $m_1 = \alpha^2 m$ $\Rightarrow$ $\bar{r}(t_1)$ satisfies $m_1 \frac{d^2\bar{r}}{dt_1^2} = -\frac{\partial U}{\partial \bar{r}}$

i.e. if $m_1 = \frac{m}{9} \rightsquigarrow t_1 = \frac{t}{3}$

Let $U(r)$ be a homogeneous f'n of degree ν

i.e. $U(\alpha r) = \alpha^{\nu} U(r)$, $\forall \alpha > 0$

<u>Exer</u>, Prove: if γ is an orbit, then $\alpha\gamma$ is an orbit. What is the ratio of circulation times?

Kepler <u>III</u> law $\nu = -1$ Find $t_1 = f(t)$

Hooke's law $\nu = 2$ Derive $t_1 = t$

<u>Similarity</u>

1. Kid's problem: Canada is 5,500 km
map is 55 cm

scale is $1\text{ cm} = 100\text{ km} = 100 \cdot 10^3 \cdot 10^2 = 1:10^7$

Then: Canada ~ 40 mln population $: 10^7$
$= 4$ persons

should be able to stand on a map
$1:10^7$. Catch: Areas: $1 : (10^7)^2$

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2. What is faster: skin a bucket of large or small potatoes (of the same weight)

weight $\sim$ size$^3 = L^3$ (skin $\sim$ size$^2 = \sqrt[3]{M^2}$)
skin $\sim$ size$^2 = L^2$

# of potatoes in a bucket $n \sim \frac{1}{L^3}$. total skin area $\sim$

$$\sim n \cdot L^2 = \frac{L^2}{L^3} = \frac{1}{L}$$

larger $L \Rightarrow$ less skin

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3. A desert animal has to cover great distances w/o water. How does the max time the animal can run depend on its size L?

Sol'n The stored water $\sim$ volume of body $\sim L^3$

evaporation $\sim$ surface $= L^2 \Rightarrow$ the max time $\sim L$

Dimensional consideration $\underset{\text{period}}{T} = f(r, m, F)$

Physical units ~ length, mass, time (L, M, t)

$r \sim L$, $m \sim M$, $F = ma \sim ML/t^2$

$\Rightarrow T \sim \left(\frac{mr}{F}\right)^{1/2}$. Newton noted:

for $F = G\frac{m_1 m_2}{r^\alpha}$ with unknown α

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{m_1 r_1}{F_1}\right) \Big/ \left(\frac{m_2 r_2}{F_2}\right) = \left(\frac{\cancel{m_1} r_1}{\cancel{m_1} \cancel{M} / r_1^\alpha}\right) \Big/ \left(\frac{\cancel{m_2} r_2}{\cancel{m_2} \cancel{M} / r_2^\alpha}\right) = \left(\frac{r_1}{r_2}\right)^{\alpha+1}$$

Thus only $\alpha = 2$ agrees w/ III Kepler's law!

Return to conserv. fields in 3-space

$$\ddot{\bar r} = -\frac{\partial U}{\partial \bar r}, \quad \bar r \in \mathbb{R}^3, \quad E = \frac{\dot{\bar r}^2}{2} + U(\bar r)$$

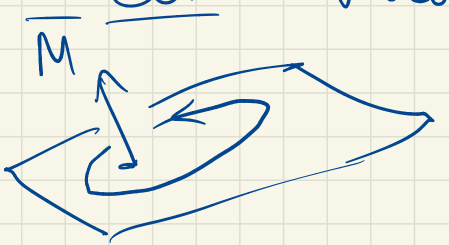
Conserv. of energy $E = \text{const}$

1) central fields $U = U(r)$, $r = |\bar r|$

$\bar M := \bar r \times \dot{\bar r}$ angular momentum w.r.t. O. Then $\frac{d\bar M}{dt} = (r \times \dot r)^{\cdot} = 0$

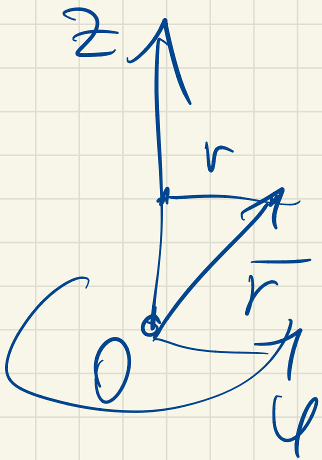
$\bar M = \text{const}$

Cor $\forall$ orbits are planar and $\perp$ to $\bar M$



Indeed, $(\bar M, \bar r) = (\bar r \times \dot{\bar r}, \bar r) = 0$

Axially symmetric fields



= invariant w.r.t. rotations about fixed axis

Then for conservative fields $\Leftarrow$

$$U = U(r, z)$$ (doesn't dep on φ)

z, r, φ – cylindrical coord's