Lecture 6

2026

Thm 2 (Bohlin) Assume $w \in \mathbb{C}$ moves according to Hooke $\ddot{w} = -w$. Let $z = w^2$, and new time τ is such that the law of sectorial area is satisfied. Then $z(\tau)$ satisfies Kepler: $\dfrac{d^2 z}{d\tau^2} = -c\dfrac{z}{|z|^3}$

Pf The law of areas: $\dfrac{|w|^2 d\varphi}{dt} = \text{const}$, $\dfrac{|z|^2 d\varphi}{d\tau} = \text{const}$

Set $\dfrac{d\tau}{dt} = \dfrac{|z|^2}{|w|^2} \Rightarrow \dfrac{d}{d\tau} = \dfrac{|w|^2}{|z|^2}\dfrac{d}{dt} = \dfrac{1}{|w|^2}\dfrac{d}{dt}$ and

$$\frac{d^2 z}{d\tau^2} = \frac{1}{w\bar{w}}\frac{d}{dt}\left(\frac{1}{w\bar{w}}\frac{dw^2}{dt}\right) = \frac{2}{w\bar{w}}\frac{d}{dt}\left(\frac{1}{\bar{w}}\frac{dw}{dt}\right)$$

$$= -\frac{2}{w\bar{w}}\left(\frac{1}{\bar{w}^2}\frac{dw}{dt}\frac{d\bar{w}}{dt} + \frac{w}{\bar{w}}\right) = -2w^{-1}\bar{w}^{-3}\left(|\dot{w}|^2 + |w|^2\right)$$

$$= -4E\, w^{-1}\bar{w}^{-3}$$

used $\ddot{w} = -w$

Here $|\dot{w}|^2 + |w|^2 = 2E$ on the trajectory: conserv. of energy

Since $\dfrac{z}{|z|^3} = \dfrac{w^2}{|w^2|^3} = \dfrac{w^2}{w^3 \bar{w}^3} = w^{-1} \bar{w}^{-3}$,

$$\Rightarrow \quad \frac{d^2 z}{d\tau^2} = -c \frac{z}{|z|^3} \quad \text{for } c = 4E \qquad \blacksquare$$

<u>Cor</u>. Thm 1+2 $\Rightarrow$ Kepler's bounded orbits are ellipses w/ foci at 0 (i.e. for $E<0$)

<u>Thm 3</u> (Exer) Trajectories for force $\sim |w|^a$ are taken by $\boxed{z = w^\alpha}$ for $\alpha = \dfrac{a+3}{2}$ into trajectories

w/ force $\sim |z|^A$ (Find the relation of a and A)

Here $\underset{\text{Hooke}}{a=1}$, $\alpha = \frac{a+3}{2} = \frac{1+3}{2} = 2$, $z = w^2$

$\underset{\text{Kepler}}{A=-2}$

Similarity in physics

Observation: Let $\bar{r}(t)$ satisfies $m\frac{d^2\bar{r}}{dt^2} = -\frac{\partial U}{\partial \bar{r}}$

take $t_1 = \alpha t$, $m_1 = \alpha^2 m$ $\Rightarrow$ $\bar{r}(t_1)$ satisfies $m_1\frac{d^2\bar{r}}{dt_1^2} = -\frac{\partial U}{\partial \bar{r}}$

i.e. if $m_1 = \frac{m}{9}$ $\rightsquigarrow$ $t_1 = \frac{t}{3}$ (3 times as fast)

Let $U(r)$ be a homogeneous f'n of degree ν

i.e. $$U(\alpha r) = \alpha^{\nu} U(r) \qquad \forall \alpha > 0$$

Exer
Prove

If γ is an orbit, then $\alpha\gamma$ is an orbit

What is the ratio of circulation times?

Derive $t_1 = t$ for Hooke's law $\nu = 2$, $U \sim r^2$

Find $t_1 = f(t)$ for $\nu = -1$, $U \sim \frac{1}{r}$

(obtain Kepler's III law)

Similarity

1. Kid's problem. Canada is 5,500 km wide

map is 55 cm

scale is $1\,cm = 100\,km = \underset{m}{100 \cdot 10^3} \cdot \underset{cm}{10^2} = 1 : 10^7$

Then ~ 40 mln population $: 10^7 = 4$ persons should stay on a map $1 : 10^7$. Catch? Areas $\sim (10^7)^2$

2. What is faster: to skin a bucket of large or small potatoes? (of the same weight)

Fixed M, weight of 1 potat. $\sim$ size$^3 = L^3$

L - size

skin area $\sim$ size$^2 = L^2$

\# of potatoes in a bucket $n \sim \frac{1}{L^3} \Rightarrow$ total skin area $\sim n \cdot L^2 = \frac{L^2}{L^3} = \frac{1}{L}$

better - larger L $\Rightarrow$ less skin

3. A desert animal has to cover great distances w/o water. How does the max time the animal can run depend on its size L?

Sol'n The storage of water $\sim$ volume of body $\sim L^3$

evaporation $\sim$ surface $= L^2$ $\Rightarrow$ the max time $\sim L$. ▨

Dimensional consideration $T = f(r, m, F)$

Physical units (following Maxwell) –

length, mass, time

L M t

$r \sim L, \quad m \sim M, \quad F = ma \sim M \cdot L / t^2$

$\Rightarrow$ period $T \sim \left(\dfrac{m r}{F}\right)^{1/2}$. Newton noted:

For $F = G \dfrac{m_1 m_2}{r^\alpha}$ with unknown α

$$\left(\frac{T_1}{T_2}\right)^2 = \frac{\left(\dfrac{m_1 r_1}{F_1}\right)}{\left(\dfrac{m_2 r_2}{F_2}\right)} = \frac{\left(\dfrac{\cancel{m_1} r_1}{\left(\dfrac{\cancel{m_1 M}}{r_1^\alpha}\right)}\right)}{\left(\dfrac{\cancel{m_2} r_2}{\left(\dfrac{\cancel{m_2 M}}{r_2^\alpha}\right)}\right)} = \frac{r_1^{\alpha+1}}{r_2^{\alpha+1}} = \left(\frac{r_1}{r_2}\right)^{\alpha+1}$$

Thus only $\alpha = 2$ agrees w/ Kepler's <u>III</u> law!