Today we will continue discussing limit laws.

Homework before Thursday’s class: watch videos 2.14, 2.15.
True or False?

Is this theorem true?

Claim

Let \( a \in \mathbb{R} \).
Let \( f \) and \( g \) be functions defined near \( a \).

- IF \( \lim_{x \to a} f(x) = 0 \),
- THEN \( \lim_{x \to a} [f(x)g(x)] = 0 \).
Bounded functions

Definition

A function \( f(x) \) is **bounded** on a set \( I \) when

\[
\exists M \in \mathbb{R} \text{ such that } \forall x \in I, \ |f(x)| \leq M.
\]

Are these functions bounded?

- \( f(x) = x \) on the set \( \mathbb{R} \).
- \( f(x) = x \) on the set \((0, 1)\).
- \( f(x) = \sin x \) on the set \( \mathbb{R} \).
- \( f(x) = 1/x \) on the set \((0, 1)\).
A kind of product law

**Theorem**

Let \( f \) and \( g \) be functions defined on \( \mathbb{R} \).
Assume that \( \lim_{x \to a} f(x) = 0 \), and \( g \) is bounded on \( \mathbb{R} \).

**THEN** \( \lim_{x \to a} [f(x)g(x)] = 0 \)

Example:

\[
\lim_{x \to 0} x^2 \sin \left( \frac{x + 7}{x^3} \right) = 0
\]
A kind of product law

**Theorem**

Let $f$ and $g$ be functions defined on $\mathbb{R}$. Assume that $\lim_{x \to a} f(x) = 0$, and $g$ is bounded on $\mathbb{R}$. Then $\lim_{x \to a} [f(x)g(x)] = 0$

Example:

$$\lim_{x \to 0} x^2 \sin \left( \frac{x + 7}{x^3} \right) = 0$$

Write down a formal proof.
Since $g$ is bounded, $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$

Since $\lim_{{x \to a}} f(x) = 0$, there exists $\delta_1 > 0$ s.t. if $0 < |x - a| < \delta_1$, then

$$|f(x) - 0| = |f(x)| < \varepsilon_1 = \frac{\varepsilon}{M}.$$

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$$

In summary, by setting $\delta = \min\{\delta_1\}$, we find that if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$. 
Second student’s proof

- WTS: \( \forall \varepsilon > 0, \exists \delta > 0 \) s.t. \( 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon \).

- Let \( \varepsilon > 0 \).

- We know \( \lim_{x \to a} f(x) = 0 \). In this definition, let \( \varepsilon_1 = \frac{\varepsilon}{M} \).

- We know \( \exists \delta_1 \in \mathbb{R} \) s.t. \( 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1 = \frac{\varepsilon}{M} \).

- Assume \( 0 < |x - a| < \delta \)

- Since \( \exists M > 0 \) s.t. \( \forall x \neq 0, |g(x)| \leq M \)

\[ |f(x)g(x)| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon. \]
Adjusted second student’s proof

- **WTS:** \( \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \epsilon. \)

- Since \( g \) is bounded, \( \exists M > 0 \text{ s.t. } \forall x \in \mathbb{R}, |g(x)| \leq M \)

- Let \( \epsilon > 0. \)

- We know \( \lim_{x \to a} f(x) = 0. \) In this definition, let \( \epsilon_1 = \frac{\epsilon}{M}. \)

- Then \( \exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \epsilon_1 = \frac{\epsilon}{M}. \)

- Set \( \delta = \delta_1 \) and assume \( 0 < |x - a| < \delta \)

- Then for all such \( x \) we have

  \[
  |f(x)g(x)| = |f(x)| \cdot |g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon.
  \]

- This proves that \( \lim_{x \to a} f(x)g(x) = 0. \)
The Squeeze Theorem

Prove that

$$\lim_{x \to 0} x^2 \sin \left( \frac{x + 7}{x^3} \right) = 0$$

from the Squeeze Theorem.