MAT137

- Today we will continue discussing limit laws.
- Homework before Thursday's class: watch videos 2.14, 2.15.

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

Bounded functions

Definition

A function $f(x)$ is **bounded** on a set I when

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in I,\ |f(x)| \leq M.$$

Are these functions bounded?

- $f(x) = x$ on the set $\mathbb{R}$.
- $f(x) = x$ on the set $(0, 1)$.
- $f(x) = \sin x$ on the set $\mathbb{R}$.
- $f(x) = 1/x$ on the set $(0, 1)$.

A kind of product law

Theorem

Let f and g be functions defined on $\mathbb{R}$.
Assume that $\lim_{x \to a} f(x) = 0$, and g is bounded on $\mathbb{R}$.
THEN $\lim_{x \to a} [f(x)g(x)] = 0$

Example:

$$\lim_{x \to 0} x^2 \sin\left(\frac{x+7}{x^3}\right) = 0$$

A kind of product law

Theorem

Let f and g be functions defined on $\mathbb{R}$.
Assume that $\lim_{x \to a} f(x) = 0$, and g is bounded on $\mathbb{R}$.
THEN $\lim_{x \to a} [f(x)g(x)] = 0$

Example:

$$\lim_{x \to 0} x^2 \sin\left(\frac{x+7}{x^3}\right) = 0$$

Write down a formal proof.

First student's proof

- Since g is bounded, $\exists M > 0$ s.t.$\forall x \neq 0, |g(x)| \leq M$
- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t. if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.

Second student's proof

- WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- Let $\varepsilon > 0$.
- We know $\lim_{x \to a} f(x) = 0$. In this definition, let $\varepsilon_1 = \dfrac{\varepsilon}{M}$.
- We know $\exists \delta_1 \in \mathbb{R}$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.
- Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
  $|f(x)g(x)| \leq \dfrac{\varepsilon}{M} \cdot M = \varepsilon$.

Adjusted second student's proof

- WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- Since g is bounded, $\exists M > 0$ s.t.$\forall x \in \mathbb{R}, |g(x)| \leq M$
- Let $\varepsilon > 0$.
- We know $\lim\limits_{x \to a} f(x) = 0$. In this definition, let $\varepsilon_1 = \dfrac{\varepsilon}{M}$.
- Then $\exists \delta_1 \in \mathbb{R}$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.
- Set $\delta = \delta_1$ and assume $0 < |x - a| < \delta$
- Then for all such x we have

  $|f(x)g(x)| = |f(x)| \cdot |g(x)| < \dfrac{\varepsilon}{M} \cdot M = \varepsilon.$
- This proves that $\lim\limits_{x \to a} f(x)g(x) = 0$.

The Squeeze Theorem

Prove that

$$\lim_{x \to 0} x^2 \sin\left(\frac{x+7}{x^3}\right) = 0$$

from the Squeeze Theorem.