- Today we will discuss limit laws.

- Homework before Wednesday's class: watch videos 2.12, 2.13.

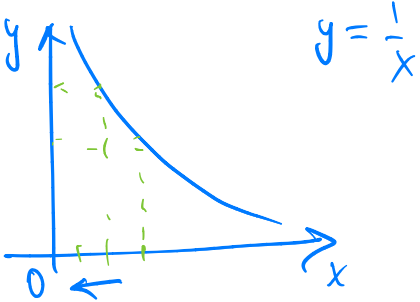
Proof of non-existence

Goal

We want to prove that

$$\lim_{x \to 0^+} \frac{1}{x} \text{ does not exist} \tag{1}$$

directly from the definition.



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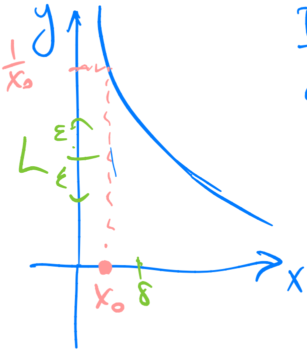
directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

Pf. WTS $\forall L \in \mathbb{R}\ \exists \varepsilon > 0$ s.t. $\forall \delta > 0$

$\exists x_0$ s.t. $0 < x_0 < \delta$, but $\left|\frac{1}{x_0} - L\right| \geq \varepsilon$



$|a| \geq a$

Indeed, take any $L > 0$ and $\varepsilon > 0$ (consider $L \leq 0$ later).

Rough work: need to find such an x_0 that

$$\left|\frac{1}{x_0} - L\right| \geq \varepsilon$$

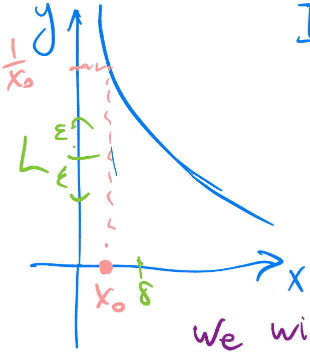
$$\Uparrow$$

$$\frac{1}{x_0} - L \geq \varepsilon$$

$$\frac{1}{x_0} \geq \underline{L + \varepsilon} > 0$$

Pf. WTS $\forall L \in \mathbb{R}\ \exists \varepsilon > 0$ s.t. $\forall \delta > 0\ \exists x_0$ s.t.

$$0 < x_0 < \delta, \text{ but } \left|\frac{1}{x_0} - L\right| \geq \varepsilon$$



Indeed, take any $L > 0$ (consider $L \leq 0$ similarly yourself) and take $\varepsilon = 1$.

Rough work: we need to find such an x_0 that $\left|\frac{1}{x_0} - L\right| \geq 1$

We will solve a "more difficult" problem: find x_0 such that $\frac{1}{x_0} - L \geq 1$ or, equivalently, $\frac{1}{x_0} \geq L + 1$. Note that $L + 1 > 0$

Since $x_0 > 0$, it suffices to take $0 < x_0 \leq \frac{1}{L+1}$

Also given any $\delta > 0$ we need $0 < x_0 < \delta$

Clean proof: Take any $L > 0$, any $\delta > 0$ and $\varepsilon = 1$. Then take $x_0 \in \mathbb{R}$ such that

$$0 < x_0 < \min\left(\delta, \frac{1}{L+1}\right)$$

Then for this x_0 we have $0 < x_0 < \delta$ and $\frac{1}{x_0} > L+1$, and hence $\left|\frac{1}{x_0} - L\right| \geq 1$

This means that $\nexists \lim_{x \to 0^+} \frac{1}{x}$ $\blacksquare$

Uniqueness of limits

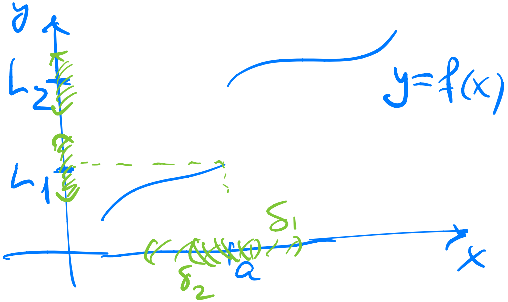
Theorem

Let a, L_1, L_2 be real numbers. Let f be a function defined on an interval containing $a \in \mathbb{R}$, except possibly at a. Suppose that

$$\lim_{x \to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to a} f(x) = L_2$$

Then $L_1 = L_2$.

By contradiction: assume that $L_1 \neq L_2$. Then take $\varepsilon = |L_1 - L_2|/3 > 0$.
Then $\exists \delta > 0$ s.t. ...



$\varepsilon = \dfrac{|L_1 - L_2|}{3} > 0$

$\lim_{x\to a} f(x) = L_i \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta_i > 0$ s.t.

$0 < |x-a| < \delta_i \Rightarrow |f(x) - L_i| < \varepsilon = \dfrac{|L_1 - L_2|}{3}$

$i = 1, 2$

Take $\delta = \min(\delta_1, \delta_2)$

Then for all x s.t.

$0 < |x-a| < \delta \Rightarrow$

$$|f(x) - L_1| < \frac{|L_1 - L_2|}{3}$$

and $|f(x) - L_2| < \frac{|L_1 - L_2|}{3}$. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq$$

$$\leq |L_1 - f(x)| + |f(x) - L_2| <$$

Δ ineq.

$$\frac{|L_1 - L_2|}{3} + \frac{|L_1 - L_2|}{3} = \frac{2}{3}|L_1 - L_2|$$

We obtained that for those x's

$|L_1 - L_2| < \frac{2}{3}|L_1 - L_2|$, which is a contradiction. Hence $L_1 = L_2$ $\boxtimes$

Recall the limit laws:

Assume that

$\exists \lim_{x\to a} f(x) = L$, $\exists \lim_{x\to a} g(x) = M$

Then 1) $\exists \lim_{x\to a} (f+g)(x) = L+M$

2) $\exists \lim_{x\to a} f\cdot g(x) = L\cdot M$

3) If $M \neq 0$ then $\exists \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

Some strange guys

Can you find functions $f(x)$ and $g(x)$ with the following properties?

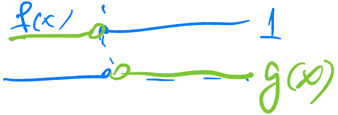
1. $\lim_{x\to 0} f(x)$ does not exists and $\lim_{x\to 0}(f(x)+g(x)) = 0$.
2. $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} f(x)g(x) = 14$.
3. $\lim_{x\to 0} f(x)$ does not exist and $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0} f(x)g(x) = 0$.
4. $\lim_{x\to 0} f(x) = 2$ and $\lim_{x\to 0} g(x) = 3$, but $\lim_{x\to 0} g(f(x)) = 14$.

1) $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x} + x$

Then $\lim_{x\to 0} f(x)$ DNE, but $\lim_{x\to 0}(f+g) = \lim_{x\to 0} x = 0$

2) $f(x) = x \qquad g(x) = \frac{14}{x}$

$\lim_{x \to 0} f(x) = 0 \qquad \lim_{x \to 0} g(x)$ DNE

3)


Note: $\lim_{x \to 0} f(x)$ DNE

$\lim_{x \to 0} g(x)$ DNE

But $\lim_{x \to 0} f \cdot g(x) = \lim_{x \to 0} 0 = 0$

4) $f(x) \equiv 2$, $\Rightarrow \lim_{x \to 0} f(x) = 2$

$g(x) = 3 + \frac{11}{2}x$, $\Rightarrow \lim_{x \to 0} g(x) = 3$, $\lim_{x \to 2} g(x) = 14$

and then $\lim_{x \to 0} g(f(x)) = \lim_{z \to 2} g(z) = 14$

Question: Can it be that $\lim_{x\to a} f(x)$ DNE,

but $\exists \lim_{x\to a} g(x)$ and $\exists \lim_{x\to a}(f(x)+g(x))$?

Let $a = 0$ in all examples and consider $\lim_{x \to 0}$

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.
Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \dfrac{f(x)}{g(x)}$?

1. The limit is 1.
2. The limit is 0.
3. The limit is ∞.
4. The limit does not exist.
5. We do not have enough information to decide.

1) $\dfrac{f}{g} = \dfrac{x}{x}$

2) $\dfrac{x^2}{x}$

3) $\dfrac{|x|}{x^2}$

4) $\dfrac{|x|}{x}$

4') $\dfrac{x \sin \frac{1}{x}}{x}$