Today: Proof of the differentiation rules.

Homework before Tuesday's class: watch videos 3.10, 3.11.
Let $a \in \mathbb{R}$.
Let $f$ be a function with domain $\mathbb{R}$.
Assume $f$ is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $f$ is continuous at $a$.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. $f'$ is continuous at $a$. 
A continuity lemma

Write a formal proof for:

**Lemma**

Let \( a \in \mathbb{R} \).
Let \( g \) be a function continuous at \( a \).
Assume that \( g(a) \neq 0 \). Then \( g(x) \neq 0 \) for \( x \) close to \( a \).

*Note:* First, figure out what “\( g(x) \neq 0 \) for \( x \) close to \( a \)” means formally.
Write a formal proof for the quotient rule for derivatives

**Theorem**

- Let \( a \in \mathbb{R} \).
- Let \( f \) and \( g \) be functions defined at and near \( a \).
  Assume \( g(a) \neq 0 \).
- We define the function \( h \) by \( h(x) = \frac{f(x)}{g(x)} \).

**If** \( f \) and \( g \) are differentiable at \( a \),

**then** \( h \) is differentiable at \( a \), and

\[
h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.
\]

Write a proof directly from the definition of derivative.

**Hint:** Imitate the proof of the product rule in Video 3.6.
Check your proof

- Are there words or only equations?
- Does every step follow logically?
- Did you only assume things you could assume?

- At some point in your proof you must have used, for example, that $g$ was continuous. (Otherwise your proof is most likely wrong.) Did you notice you were using this? Did you justify it?
Critique this proof

\[ h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \]

\[ = \lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)(x - a)} \]

\[ = \lim_{x \to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)(x - a)} \]

\[ = \lim_{x \to a} \left\{ \left[ \frac{f(x) - f(a)}{x - a}g(a) - f(a) \frac{g(x) - g(a)}{x - a} \right] \frac{1}{g(x)g(a)} \right\} \]

\[ = [f'(a)g(a) - f(a)g'(a)] \frac{1}{g(a)g(a)} \]
What is the equation for this function? \( f(x) = \ldots \)
Construct a function that has both
- a vertical tangent line at $x = 1$, and
- a vertical asymptote at $x = -1$. 
1. Sketch the graph of a continuous function whose derivative has the graph below.

2. Sketch the graph of a non-continuous function whose derivative has the graph below.