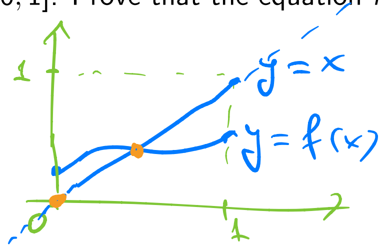
- Today: Definition of derivative.
- Homework before Wednesday class: watch videos 3.4, 3.5, 3.8.

From last time: More uses of IVT

1. Prove that the equation $x^3 + \sin x - 1 = 0$ has a solution.

2. Prove that the equation $x^4 - 2x = 100$ has at least two solutions.

3. Suppose that $f(x)$ is a continuous function on $[0, 1]$ such that $f(x) \in [0, 1]$. Prove that the equation $f(x) = x$ has a solution.



Sol: If $f(0)=0$ or $f(1)=1$, then they are sol's of the eq'n $f(x)=x$ ($x=0$ or 1 resp.)

Otherwise we know that $f(0)>0$ and $f(1)<1$

Consider the f'n $g(x)=f(x)-x$

Then $g(x)$ is contin (since $f(x)$ and x are on $[0,1]$)

$g(0)=f(0)-0>0$

$g(1)=f(1)-1<0$

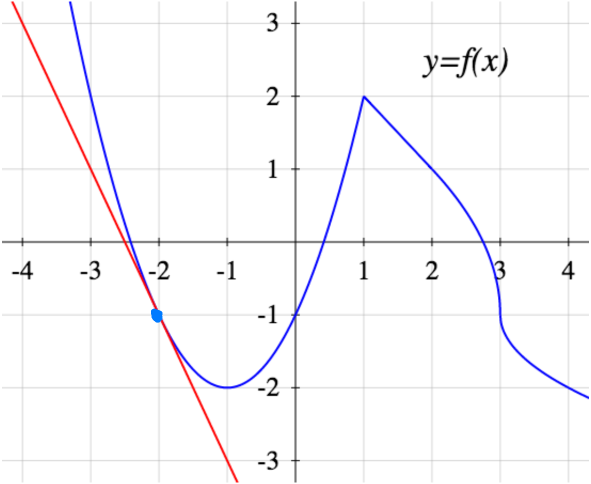
$\Rightarrow$ by the IVT $\exists\, c\in(0,1)$

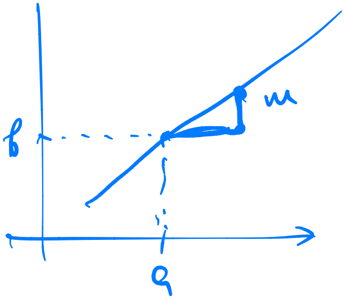
$g(c)=0 \Leftrightarrow f(c)-c=0$

$\Leftrightarrow f(c)=c$, a sol'n $\blacksquare$

Tangent line from a graph

Below is the graph of the function f. Write the equation of the line tangent to it at the point with x–coordinate -2.





$m = \text{slope}$

$$y - b = m(x-a)$$

$a = -2$

$b = -1, \quad m = -2$

$$y - (-1) = -2(x - (-2))$$

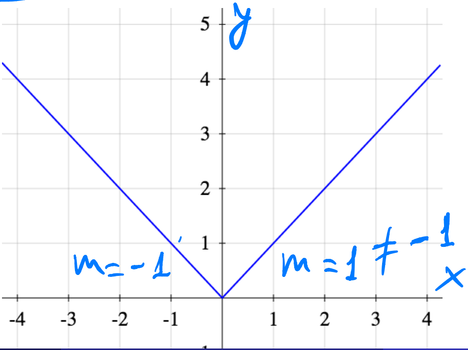
$$y + 1 = -2x - 4$$

$$y = -2x - 5$$

Absolute value and tangent lines

At (0,0) the graph of $y = |x|$...

1. ... has one tangent line: $y = 0$
2. ... has one tangent line: $x = 0$
3. ... has two tangent lines $y = x$ and $y = -x$
4. ... has no tangent line

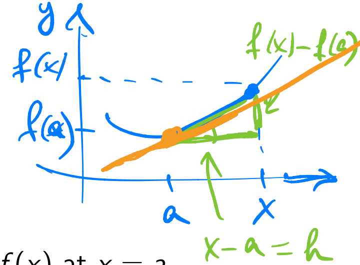


The definition

$x = a + h$

Let $f(x)$ be a function defined on some interval containing a. We say that $f(x)$ is *differentiable at a* when

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



exists. Notation for the limit: $f'(a)$.

Meaning of derivative $f'(a)$:

- the slope of the tangent to the graph $y = f(x)$ at $x = a$.
- the instantaneous rate of change of $f(x)$ at $x = a$.

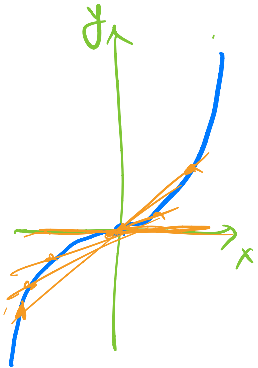
Absolute value and derivatives

Let $g(x) = x|x|$. What is $g'(0)$?

1. It is 0.
2. It does not exist because $|x|$ is not differentiable at 0.
3. It does not exist because the right- and left-limits, when computing the derivative, are different.
4. It does not exist because it has a corner.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$g(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$



$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x}$$

$$= \lim_{x \to 0^+} x = 0$$

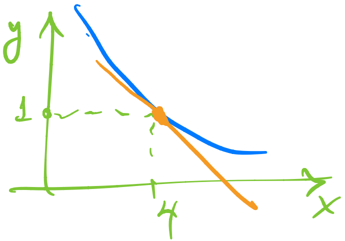
$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^2 - 0}{x} = \lim_{x \to 0^-} (-x) = 0$$

$$\Rightarrow \exists \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 0 = g'(0)$$

Let

$$g(x) = \frac{2}{\sqrt{x}}$$

Calculate $g'(4)$ directly from the definition of derivative as a limit.



$$\lim_{x\to 4}\frac{g(x)-g(4)}{x-4} = \lim_{x\to 4}\frac{\frac{2}{\sqrt{x}}-\underbrace{\frac{2}{\sqrt{4}}}_{=1}}{x-4} = \lim_{x\to 4}\frac{2-\sqrt{x}}{\sqrt{x}(x-4)}$$

$$= \lim_{x\to 4}\frac{\overbrace{(2-\sqrt{x})(2+\sqrt{x})}^{2^2-x}}{\sqrt{x}(x-4)(2+\sqrt{x})} = \lim_{x\to 4}\frac{\overbrace{\cancel{4-x}}^{=-1}}{\sqrt{x}\,\cancel{(x-4)}(2+\sqrt{x})}$$

$x>0$

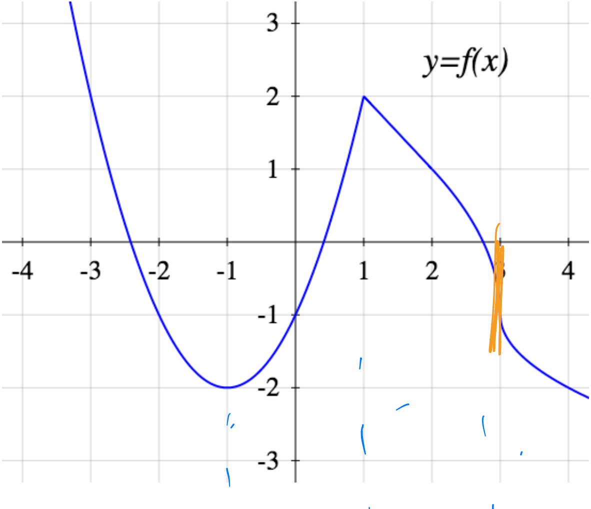
$$= \lim_{x\to 4}\frac{-1}{\sqrt{x}(2+\sqrt{x})} = -\frac{1}{\sqrt{4}(2+\sqrt{4})} = -\frac{1}{2(2+2)} = -\frac{1}{8}$$

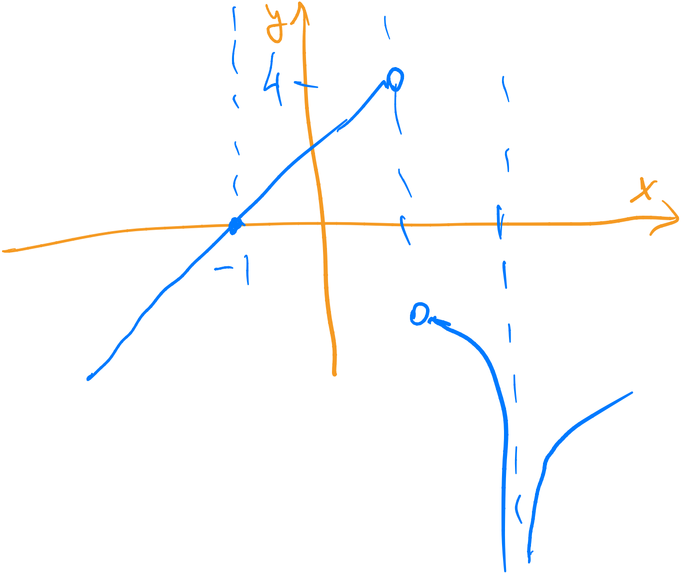
contin at $x=4$

Note: $\frac{2}{\sqrt{x}}-1 = \frac{2-\sqrt{x}}{\sqrt{x}}$

Derivative from a graph

Below is the graph of the function f.
Sketch the graph of its derivative f'.




y
x
4
-1