Today: the remarkable limit and continuity of composition.

Homework before Wednesday’s class: watch videos 2.19, 2.20.
Elementary continuous functions

\[ Y = \text{CONSTANT} \]

\[ Y = x^2 \]

\[ Y = x^3 \]

\[ Y = \cos x \]
Is it possible to construct functions such that....?

1. \( \lim_{x \to 1} f(x) = 2 \)

2. \( \lim_{u \to 2} g(u) = 3 \)

3. \( \lim_{x \to 1} g(f(x)) = 42 \)
True or False?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) = g \left( \lim_{x \to a} f(x) \right)$$
True or False?

(Assuming these limits exist)

\[
\lim_{{x \to a}} g(f(x)) = g \left( \lim_{{x \to a}} f(x) \right)
\]

What extra condition do we need to add for this to be true?
A composition theorem

Theorem
Let $a, L \in \mathbb{R}$.
Let $f$ and $g$ be functions.
IF

- $\lim_{x \to a} f(x) = L$
- $g$ is continuous at $L$

THEN $\lim_{x \to a} g(f(x)) = g(L)$

How can we prove this?
A composition theorem

Theorem

Let $a, L \in \mathbb{R}$.
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How can we prove this?

$|g(f(x)) - g(L)| < \varepsilon$
Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

*Suggestion*: If you don't know how to start, try some sample values of $x$ and $y$.

Write a similar expression to compute $\min\{x, y\}$.
We want to prove the following theorem

**Theorem**

IF $f$ and $g$ are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know. What is the fastest way to prove this?

*Hint:* There is a way to prove this quickly without writing any epsilons.
Computations!

Using that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), compute the following limits:

1. \( \lim_{x \to 2} \frac{\sin x}{x} \)
2. \( \lim_{x \to 0} \frac{\sin(5x)}{x} \)
3. \( \lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4} \)
4. \( \lim_{x \to 0} \frac{\sin e^x}{e^x} \)
5. \( \lim_{x \to 0} \frac{1 - \cos x}{x} \)
6. \( \lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)} \)
Computations!

Using that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), compute the following limits:

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5. \( \lim_{x \to 0} \frac{1 - \cos x}{x} \)

6. \( \lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)} \)

7. \( \lim_{x \to 0} \left[ (\sin x) (\cos(2x)) (\tan(3x)) (\sec(4x)) (\csc(5x)) (\cot(6x)) \right] \)
An extended version