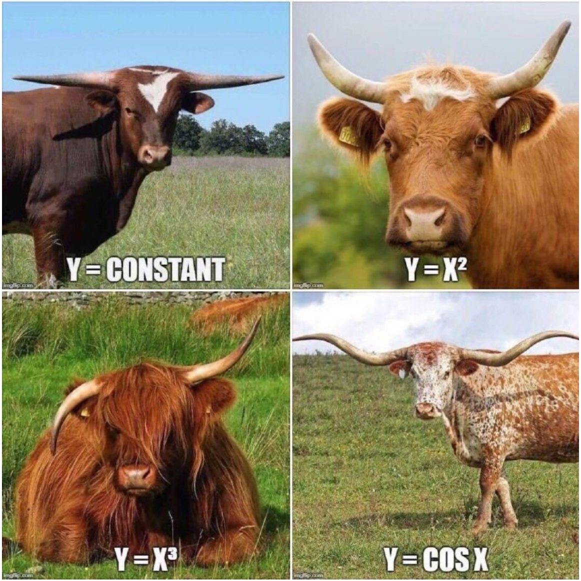
- Today: the remarkable limit and continuity of composition.
- Homework before Wednesday's class: watch videos 2.19, 2.20.

Elementary continuous functions



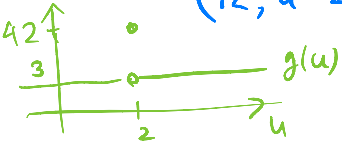
A difficult example

Take $f(x) = 2$ $\forall x \in \mathbb{R}$

$$g(u) = \begin{cases} 3, & u \neq 2 \\ 42, & u = 2 \end{cases}$$

Is it possible to construct functions such that....?

1. $\lim\limits_{x \to 1} f(x) = 2$
2. $\lim\limits_{u \to 2} g(u) = 3$
3. $\lim\limits_{x \to 1} g(f(x)) = 42$



$g(f(x)) = g(2) = 42$ $\forall x \in \mathbb{R}$

True or False?

(Assuming these limits exist)

$$\lim_{x\to a} g(f(x)) \;=\; g\left(\overset{L}{\overset{\shortparallel}{\lim_{x\to a} f(x)}}\right)$$

What extra condition do we need to add for this to be true?

g is contin at L

A composition theorem

Theorem

Let $a, L \in \mathbb{R}$.
Let f and g be functions.
IF

- $\lim_{x \to a} f(x) = L$
- g is continuous at L

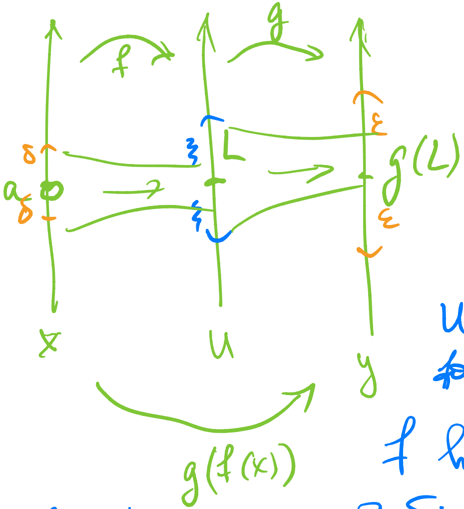
THEN $\lim_{x \to a} g(f(x)) = g(L)$

How can we prove this?

WTS $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$|g(f(x)) - g(L)| < \varepsilon$

$0 < |x - a| < \delta \Rightarrow |g(f(x)) - g(L)| < \varepsilon$



g is <u>contin</u> at L, i.e.

$\forall \varepsilon > 0 \ \exists \xi > 0$ s.t

$|u - L| < \xi \Rightarrow$

$|g(u) - g(L)| < \varepsilon$

Use this ξ as a "new ε" for function f.

f has a lim at $x = a$, i.e

for this $\xi > 0 \ \exists \delta > 0$ s.t.

$0 < |x - a| < \delta \Rightarrow |\underbrace{f(x)}_{u} - L| < \xi$

Thus for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \Rightarrow |\underbrace{f(x)}_{u} - L| < \xi$$

$$\Rightarrow |g(u) - g(L)| = |g(f(x)) - g(L)| < \varepsilon$$

$\blacksquare$

A new function

Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

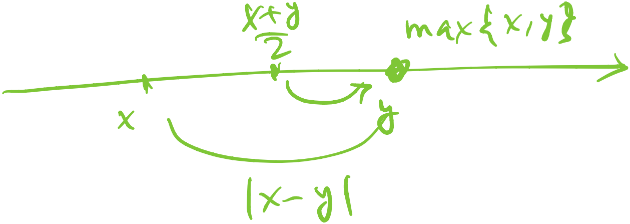
$$f(x,y) = \frac{x + y + |x - y|}{2}$$

Suggestion: If you don't know how to start, try some sample values of x and y.

Write a similar expression to compute $\min\{x, y\}$.

$x=3,\ y=5 \quad f(x,y)=\dfrac{3+5+|3-5|}{2}=\dfrac{8+2}{2}=5$

$x=7,\ y=-1,\ f(x,y)=\dfrac{7-1+|7-(-1)|}{2}=\dfrac{6+8}{2}=7$



More continuous functions

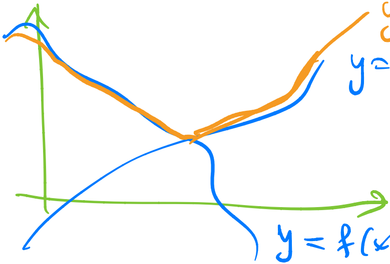
We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know.
What is the fastest way to prove this?

Hint: There is a way to prove this quickly without writing any epsilons.



Pf

$$h(x)=\max\{f(x),g(x)\}=\frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

$F(x)=|x|$ is contin on $\mathbb{R}$

$$h(x)=\frac{f(x)+g(x)+F(f(x)-g(x))}{2}$$

sums, differen. compos. of contin f's

$\Rightarrow$ h is contin. $\blacksquare$

Computations!

Using that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, compute the following limits:

$u = e^x$

1. $\lim_{x \to 2} \frac{\sin x}{x} = \frac{\sin 2}{2}$
2. $\lim_{x \to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x \to 0} \frac{\sin e^x}{e^x} = \lim_{u \to 1} \frac{\sin u}{u} = \frac{\sin 1}{1}$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x}$
6. $\lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$

The remarkable limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

2. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x} = \lim\limits_{x\to 0} \frac{\sin 5x}{5x}\cdot 5 \left(= \lim\limits_{u\to 0} \frac{\sin u}{u}\cdot 5\right)$

$5x = u$ $\qquad = 1\cdot 5 = 5$

3. $\lim\limits_{x\to 0} \frac{\tan^2(2x^2)}{x^4} = \lim\limits_{x\to 0} \frac{\sin^2(2x^2)}{\underbrace{\cos^2(2x^2)}_{\to 1}}\cdot\frac{1}{x^4}$

$= \lim\limits_{x\to 0} \frac{\sin^2(2x^2)}{(2x^2)^2}\cdot\frac{(2x^2)^2}{\cos^2(2x^2)\cdot x^4}$

$= \lim\limits_{x\to 0} \underbrace{\left(\frac{\sin(2x^2)}{2x^2}\right)^2}_{1^2}\cdot\frac{2^2\cancel{x^4}}{\underbrace{\cos^2(2x^2)}_{1^2}\cancel{x^4}} = 2^2 = 4$

5. $$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\overbrace{(1-\cos x)(1+\cos x)}^{=1-\cos^2 x}}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1^2}\cdot\underbrace{\frac{x}{1+\cos x}}_{\to 0}=0$$

6. $$\lim_{x\to 0}\frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}=\lim\underbrace{\left(\frac{\sin(2x^{20})}{2x^{20}}\right)^{10}}_{\to 1^{10}}\cdot$$

$$\cdot\underbrace{\frac{1}{\cos^{10}(2x^{20})}}_{\to 1}\cdot\underbrace{\left(\frac{3x}{\sin(3x)}\right)^{200}}_{\to 1}\cdot\frac{(2\cancel{x}^{20})^{10}}{(3\cancel{x})^{200}}=\frac{2^{10}}{3^{200}}$$

Computations!

Using that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, compute the following limits:

1. $\lim_{x \to 2} \frac{\sin x}{x}$
2. $\lim_{x \to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x \to 0} \frac{\sin e^x}{e^x}$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x}$
6. $\lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$
7. $\lim_{x \to 0} \left[(\sin x)(\cos(2x))(\tan(3x))(\sec(4x))(\csc(5x))(\cot(6x))\right]$

An extended version