Today: More indeterminate forms and L’Hôpital’s Rule

1. Prove that $\forall c \in \mathbb{R}, \exists a \in \mathbb{R}$ and functions $f$ and $g$ s.t.

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0, \quad \lim_{x \to a} \frac{f(x)}{g(x)} = c$$

This is how you show that $\frac{0}{0}$ is an indeterminate form.
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$$

This is how you show that $\frac{0}{0}$ is an indeterminate form.

2. Prove the same way that $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$ are also indeterminate forms.
1. Prove that \( \forall c \in \mathbb{R}, \exists a \in \mathbb{R} \) and functions \( f \) and \( g \) s.t.

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\]

This is how you show that \( \frac{0}{0} \) is an indeterminate form.

2. Prove the same way that \( \frac{\infty}{\infty} \), \( 0 \cdot \infty \), and \( \infty - \infty \) are also indeterminate forms.

3. Prove that \( 1^\infty \), \( 0^0 \), and \( \infty^0 \) are indeterminate forms. (You will only get \( c \geq 0 \) this time)
Compute:

1. \( \lim_{x \to 0} \left[ \frac{\csc x}{x} - \frac{\cot x}{x} \right] \)

2. \( \lim_{x \to \infty} \left[ \ln(x + 2) - \ln(3x + 4) \right] \)

3. \( \lim_{x \to 1} \left[ \frac{2}{x^2 - 1} - \frac{1}{x - 1} \right] \)

4. \( \lim_{x \to -\infty} \left[ \sqrt{x^2 + 3x} - \sqrt{x^2 - 3x} \right] \)
Compute:

1. \( \lim_{x \to 0} \left[ 1 + 2 \sin(3x) \right]^4 \cot(5x) \)

2. \( \lim_{x \to \infty} \left( \frac{x + 2}{x - 2} \right)^{3x} \)

3. \( \lim_{x \to 0^+} x^x \)

4. \( \lim_{x \to \frac{\pi}{2}^-} \left( \tan x \right)^{\cos x} \)

5. \( \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}} \)
Let $f$ be a function with domain $\mathbb{R}$. Assume $f$ is differentiable as many times as needed.

1. Find $a, b \in \mathbb{R}$ such that
\[ \lim_{x \to 0} \frac{f(x) - [a + bx]}{x} = 0 \]

2. Find $a, b, c \in \mathbb{R}$ such that
\[ \lim_{x \to 0} \frac{f(x) - [a + bx + cx^2]}{x^2} = 0 \]

3. Let $N \in \mathbb{N}$. Find a polynomial $P_N$ such that
\[ \lim_{x \to 0} \frac{f(x) - P_N(x)}{x^N} = 0 \]
Indeterminate?

Which of the following are indeterminate forms for limits? If any of them isn’t, then what is the value of such limit?

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