Test 2 is on Friday, November 30. Deadline to request accommodations: November 23

Today: Rolle’s Theorem

Homework before Wednesday’s class: watch videos 5.7, 5.8, 5.9.
We want to prove this theorem:

**Theorem 1**

Let $f$ be a differentiable function defined on an interval $I$. 

**IF** $\forall x \in I, f'(x) \neq 0$ 

**THEN** $f$ is one-to-one on $I$. 

1. Transform $[P \Rightarrow Q]$ into $[(\neg Q) \Rightarrow (\neg P)]$.

   You get an equivalent Theorem (call it "Theorem 2").

2. Write the definition of "$f$ is not one-to-one on $I". You will need it.

3. Recall the statement of Rolle's Theorem. You will need it.

4. Do some rough work if needed.

5. Write a complete proof for Theorem 2.
We want to prove this theorem:

**Theorem 1**
Let \( f \) be a differentiable function defined on an interval \( I \).
IF \( \forall x \in I, f'(x) \neq 0 \)
THEN \( f \) is one-to-one on \( I \).

1. Transform \([P \implies Q] \) into \([(\neg Q) \implies (\neg P)]\).
   You get an equivalent Theorem (call it “Theorem 2”).
   We are going to prove Theorem 2 instead.

2. Write the definition of “\( f \) is not one-to-one on \( I \)”.
   You will need it.

3. Recall the statement of Rolle’s Theorem.
   You will need it.

4. Do some rough work if needed.

5. Write a complete proof for Theorem 2.
A variant

Complete this variation on Theorem 2. Use the weakest conditions you can to make it true.

Theorem 3

Let $a < b$. Let $f$ be a function defined on $[a, b]$. IF

- (Some conditions on continuity and differentiability)
- $f$ is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$. 
Why the three hypotheses are necessary

You have proven

**Theorem 3**

Let $a < b$. Let $f$ be a function defined on $[a, b]$.

IF

1. $f$ is continuous on $[a, b]$
2. $f$ is differentiable on $(a, b)$
3. $f$ is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Give three examples to justify that each of the three hypotheses are necessary for the theorem to be true. (Graphs of the examples are enough).
Zeroes of the derivative

Construct a function $f$ that is differentiable on $\mathbb{R}$ and such that

1. $f$ has exactly 2 zeroes and $f'$ has exactly 1 zero.
2. $f$ has exactly 2 zeroes and $f'$ has exactly 2 zeroes.
3. $f$ has exactly 3 zeroes and $f'$ has exactly 1 zero.
4. $f$ has exactly 1 zero and $f'$ has infinitely many zeroes.
How many zeroes?

Let

\[ f(x) = e^x - \sin x + x^2 + 10x \]

How many zeroes does \( f \) have?
The second Theorem of Rolle

Complete statement for this theorem and prove it.

Rolle’s Theorem 2

Let \( a < b \). Let \( f \) be a function defined on \([a,b]\).

IF

- (Some conditions on continuity and derivatives)
- \( f(a) = f'(a) = 0 \)
- \( f(b) = 0 \)

THEN \( \exists c \in (a, b) \) such that \( f''(c) = 0 \).

**Hint:** Apply the 1st Rolle’s Theorem to \( f \), then do something else.
The $N$-th Theorem of Rolle

Complete the statement for this theorem and prove it.

**Rolle’s Theorem $N$**

Let $N$ be a positive integer.
Let $a < b$. Let $f$ be a function defined on $[a, b]$.
IF
- (Some conditions on continuity and derivatives)
- (Some conditions at $a$)
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f^{(N)}(c) = 0$. 