- Today: Properties of series.

- Homework before Tuesday's class: watch videos 13.10, 13.12, as well as 13.11.

Rapid question: Convergent or divergent?

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$
3. $\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$

Rapid question: Convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$
4. $\displaystyle\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$
5. $\displaystyle\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

True or False – The Necessary Condition

1. IF $\lim_{n\to\infty} a_n = 0$, THEN $\sum_n^\infty a_n$ is convergent.
2. IF $\lim_{n\to\infty} a_n \neq 0$, THEN $\sum_n^\infty a_n$ is divergent.
3. IF $\sum_n^\infty a_n$ is convergent THEN $\lim_{n\to\infty} a_n = 0$.
4. IF $\sum_n^\infty a_n$ is divergent THEN $\lim_{n\to\infty} a_n \neq 0$.

Are all decimal expansions well-defined?

We had defined a real number as "any number with a decimal expansion". Now we understand what it means to write a number with an infinite decimal expansion. It is a series!

$$0.a_1a_2a_3a_4a_5\cdots = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \dots$$

for any "digits" a_1, a_2, a_3, ...

But this raises a question: will these series always be convergent, no matter which infinite string of digits we choose?

Yes, they will! Prove it.
(Hint: all the terms in the series are positive.)

True or False – Series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

5. IF $\lim_{n\to\infty} S_{2n}$ exists, THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

6. IF $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} a_n = 0$, THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

7. IF $\sum_{n=0}^{\infty} a_n$ is convergent, THEN $\lim_{k\to\infty} \left[\sum_{n=k}^{\infty} a_n \right] = 0$.

8. IF $\sum_{n=0}^{\infty} a_{2n}$ is convergent and $\sum_{n=0}^{\infty} a_{2n+1}$ is convergent,

   THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

Recall: Series expansion

From the geometric series, we know that when $|x| < 1$, we can expand the function $f(x) = \dfrac{1}{1-x}$ as a series:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find similar ways to write the following functions as series:

1. $g(x) = \dfrac{1}{1+x}$
2. $h(x) = \dfrac{1}{1-x^2}$
3. $k(x) = \dfrac{1}{2-x}$

Challenge

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

Hints:

1. Compute $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$
2. Compute $\displaystyle\frac{d}{dx}\left[\arctan x\right]$
3. Pretend you can take derivatives and antiderivatives of series the way you can take them of sums. Which series adds up to $\arctan x$?
4. Now attempt the original problem.