- Today: The Limit Comparison Test (LCT) for integrals.

- Homework before Tuesday's class: we are starting series, watch videos 13.2, 13.3, 13.4, as well as 13.1.

Quick review

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$
2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$
3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

A variation on LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and $L > 0$.
- THEN $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are both convergent or both divergent.

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2. Prove it.

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2. Prove it.

Hint: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, what is larger $f(x)$ or $g(x)$?

Convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$
3. $\displaystyle\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$
4. $\displaystyle\int_0^1 \cot x\,dx$
5. $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$
6. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
7. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

Harder example

For which values of $a > 0$ is the integral

$$\int_0^\infty \frac{\arctan x}{x^a}\, dx$$

convergent?

Comparison tests for type 2 improper integrals

- A type-1 improper integral is an integral of the form $\int_a^\infty f(x)dx$, where f is a continuous, bounded function on $[c, \infty)$
- A type-2 improper integral is an integral of the form $\int_a^b f(x)dx$, where f is a continuous function on $(a, b]$ possibly with a vertical asymptote at a.

In the videos, you learned BCT and LCT for type-1 improper integrals.

1. Write the statement of the BCT for type-2 improper integrals.
2. Write the statement of the LCT for type-2 improper integrals.
3. Construct an example that you can prove is convergent thanks to one of these theorems.
4. Construct an example that you can prove is divergent thanks to one of these theorems.

Torricelli's trumpet (or Gabriel's horn)

Consider the solid of revolution obtained by rotating the region under the graph of $y = 1/x$ for $x \in [1, \infty)$. Its volume is

$$V = \pi \int_1^\infty f^2(x)\, dx = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = ?$$

Its surface area is

$$S = 2\pi \int_1^\infty f(x)\sqrt{1 + (f'(x))^2} dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx$$

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Friday's puzzle – explain Painter's paradox: Since the horn has finite volume but infinite surface area, it could be filled with a finite quantity of paint and yet that paint would not be sufficient to coat its surface!