MAT137

- Today: Analytic functions.

- Homework before Thursday's class: watch videos 14.9, 14.10.

sin is analytic

Let $f(x) = \sin x$. You know its Maclaurin series is

$$S(x) \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

As you know, in order to prove that $\sin x = S(x)$ we need to show that

$$\forall x \in \mathbb{R}, \quad \lim_{n\to\infty} R_n(x) = 0$$

Use Lagrange's Remainder Theorem to prove it!

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Reminder: Lagrange's Remainder Theorem says that given f, a, x,and n with certain conditions,

$$\exists \xi \text{ between } a \text{ and } x \text{ s.t.} \qquad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

The binomial series

Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$.

1. Find a formula for its derivatives $f^{(n)}(x)$.
2. Write its Maclaurin series at 0. Call it $S(x)$.
3. What is special about this series when $\alpha \in \mathbb{N}$?
4. Now assume $\alpha \notin \mathbb{N}$. Calculate the radius of convergence of the series $S(x)$.

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5. Using Lagrange's Remainder Theorem, prove that $f(x) = S(x)$ for $0 \leq x < 1$.
   Note: It is possible to prove it for $-1 < x < 1$, but it is harder.

A C^∞ but not analytic function

Consider the function $F(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

1. Prove that, for every $n \in \mathbb{N}$, $\lim\limits_{t \to \infty} t^n e^{-t} = 0$.
2. Prove that, for every $n \in \mathbb{N}$, $\lim\limits_{x \to 0^+} \dfrac{e^{-1/x}}{x^n} = 0$.
3. Calculate $F'(x)$ for $x > 0$.
4. Calculate $F'(x)$ for $x < 0$.
5. Calculate $F'(0)$ from the definition.
6. Calculate $F''(0)$ from the definition.
7. Prove that for every $n \in \mathbb{N}$, $F^{(n)}(0) = 0$.
8. Write the Maclaurin series for F at 0.
9. Is F analytic? Is F $C^{(\infty)}$?

From before: More power series

Write the following functions as power series centered at 0:

1. $g(x) = \ln(1 + x)$
2. $h(x) = \arctan x$
3. $f(x) = \sin^2 x$.

Hint: For each of the first two, take one single derivative. Then stop to think.