Today: Ratio test.

Homework before Wednesday’s class: watch videos 14.1, 14.2.
Use Ratio test to decide which series are convergent:

1. \( \sum_{n=1}^{\infty} \frac{3^n}{n!} \)

2. \( \sum_{n=1}^{\infty} \frac{(2n)!}{n!^2 3^{n+1}} \)

3. \( \sum_{n=2}^{\infty} \frac{1}{\ln n} \)

4. \( \sum_{n=2}^{\infty} \frac{n!}{n^n} \)
Here is a new convergence test

**Theorem**

Let \( \sum a_n \) be a series. Assume the limit \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \) exists.

- IF \( 0 \leq L < 1 \) THEN the series is **???**
- IF \( L > 1 \) THEN the series is **???**

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.

*Hint:* Imitate the explanation on Video 13.18 for the Ratio Test. For large values of \( n \), what is \( |a_n| \) approximately?
Back to your mission: prove ...

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots \\
= \ln 2
\]

\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots \\
= \frac{3}{2} \ln 2
\]
STEP 2: Writing other sums in terms of harmonic sums

Recall: $H_N = \sum_{n=1}^{N} \frac{1}{n}$. It is called a harmonic sum.

Write the following sums in terms of harmonic sums.

1. $E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2N}$

2. $O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2N - 1}$

3. $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N}$
STEP 3: Finish them!

- You proved there exists a convergent sequence \( \{ c_N \}_{N=1}^{\infty} \) such that
  \[
  \forall N \in \mathbb{N}, \quad H_N = \ln N + c_N
  \]

- You wrote \( A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N} \) in terms of harmonic sums.

Calculate

\[
A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]
STEP 3: Finish them!

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in terms of harmonic sums.

Calculate

\[
A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]

**Challenge!** Calculate

\[
B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots
\]