- Today: Ratio test.
- Homework before Wednesday's class: watch videos 14.1, 14.2.

Ratio test: Convergent or divergent?

Use Ratio test to decide which series are convergent:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2\, 3^{n+1}}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{n^n}$

Root test

Here is a new convergence test

Theorem

Let $\sum_n a_n$ be a series. Assume the limit $L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$ exists.

- IF $0 \leq L < 1$ THEN the series is ???
- IF $L > 1$ THEN the series is ???

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.

Hint: Imitate the explanation on Video 13.18 for the Ratio Test.
For large values of n, what is $|a_n|$ approximately?

Back to your mission: prove ...

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$$
$$= \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots$$
$$= \frac{3}{2} \ln 2$$

STEP 2: Writing other sums in terms of harmonic sums

Recall: $H_N = \sum_{n=1}^{N} \frac{1}{n}$. It is called a harmonic sum.

Write the following sums in terms of harmonic sums.

1. $E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2N}$
2. $O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2N-1}$
3. $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N}$

STEP 3: Finish them!

- You proved there exists a convergent sequence $\{c_N\}_{N=1}^{\infty}$ such that

$$\forall N \in \mathbb{N}, \quad H_N = \ln N + c_N$$

- You wrote $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N}$ in terms of harmonic sums.

Calculate

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots$$

STEP 3: Finish them!

- You proved there exists a convergent sequence $\{c_N\}_{N=1}^{\infty}$ such that

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- You wrote $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N}$ in terms of harmonic sums.

Calculate

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots$$

Challenge! Calculate

$$B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots$$