Today: Ratio test.

Homework before Wednesday’s class: watch videos 14.1, 14.2.
Use Ratio test to decide which series are convergent:

1. \[ \sum_{n=1}^{\infty} \frac{3^n}{n!} \]

2. \[ \sum_{n=1}^{\infty} \frac{(2n)!}{n!^2 3^{n+1}} \]

3. \[ \sum_{n=2}^{\infty} \frac{1}{\ln n} \]

4. \[ \sum_{n=2}^{\infty} \frac{n!}{n^n} \]
Here is a new convergence test

**Theorem**

Let $\sum_{n} a_n$ be a series. Assume the limit $L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$ exists.

- IF $0 \leq L < 1$ THEN the series is **convergent**.
- IF $L > 1$ THEN the series is **divergent**.

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.

*Hint:* Imitate the explanation on Video 13.18 for the Ration Test. For large values of $n$, what is $|a_n|$ approximately?
Back to your mission: prove ...

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]

\[= \ln 2\]

\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots
\]

\[= \frac{3}{2} \ln 2\]
STEP 2: Writing other sums in terms of harmonic sums

Recall: \( H_N = \sum_{n=1}^{N} \frac{1}{n} \). It is called a harmonic sum.

Write the following sums in terms of harmonic sums.

1. \( E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2N} \)

2. \( O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2N-1} \)

3. \( A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N} \)
You proved there exists a convergence sequence \( \{ c_N \}_{N=1}^{\infty} \) such that

\[
\forall N \in \mathbb{N}, \quad H_N = \ln N + c_N
\]

You wrote

\[
A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N}
\]

in terms of harmonic sums.

Calculate

\[
A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]
STEP 3: Finish them!

- You proved there exists a convergence sequence \( \{c_N\}_{N=1}^{\infty} \) such that
  \[
  \forall N \in \mathbb{N}, \quad H_N = \ln N + c_N
  \]

- You wrote \( A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2N} \)
in terms of harmonic sums.

---

Calculate

\[
A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]

Challenge! Calculate

\[
B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots
\]