- Today: Absolute and conditional convergence.
- Homework before Tuesday's class: watch videos 13.18, 13.19.

Rapid questions: Convergent or divergent?

1. $\sum_{n}^{\infty} (1.1)^n$
2. $\sum_{n}^{\infty} (0.9)^n$
3. $\sum_{n}^{\infty} \frac{(-1)^n}{\ln n}$
4. $\sum_{n}^{\infty} \frac{(-1)^n}{e^{1/n}}$

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4. $\sum_{n}^{\infty}\frac{(-1)^n}{e^{1/n}}$
5. $\sum_{n}^{\infty}\frac{1}{n^{1.1}}$
6. $\sum_{n}^{\infty}\frac{1}{n^{0.9}}$
7. $\sum_{n}^{\infty}\frac{n^3+n^2+11}{n^4+2n-3}$
8. $\sum_{n}^{\infty}\frac{\sqrt{n^5+2n+16}}{n^4-11n+7}$

True or False – Absolute Values

1. IF $\{a_n\}_{n=1}^{\infty}$ is convergent, THEN $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent.
2. IF $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent, THEN $\{a_n\}_{n=1}^{\infty}$ is convergent.
3. IF $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.
4. IF $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Positive and negative terms

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
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IF $\sum$(P.T.) is...	AND $\sum$(N.T.) is...	THEN $\sum a_n$ may be...
CONV	CONV	
∞	CONV	
CONV	$-\infty$	
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	$\sum$(P.T.) may be...	$\sum$(N.T.) may be...
In general		
If $\sum a_n$ is CONV		
If $\sum \lvert a_n \rvert$ is CONV		
If $\sum a_n$ is ABS CONV		
If $\sum a_n$ is COND CONV		
If $\sum a_n = \infty$		
If $\sum a_n$ is DIV oscillating		

Challenge (from previous slides)

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

Hints:

1. Compute $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
2. Compute $\frac{d}{dx}[\arctan x]$
3. Pretend you can take derivatives and antiderivatives of series the way you can take them of sums. Which series adds up to $\arctan x$?
4. Now attempt the original problem.

Your mission: prove ...

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\dots$$
$$=\ln 2$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8}+\dots$$
$$=\frac{3}{2}\ln 2$$

STEP 1: How quickly do harmonic sums grow?

Let us call $H_n = \sum_{k=1}^{n} \frac{1}{k}$. It is called a *harmonic sum*.

This is just notation for a quantity that appears often, like $n!$:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 \qquad\qquad H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$\{H_n\}_{n=1}^{\infty}$ is a new sequence to add to our toolkit, like $\{\ln n\}_{n=1}^{\infty}$ or $\{n!\}_{n=1}^{\infty}$.

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You are going to prove

Theorem *There exists a convergent sequence* $\{c_n\}_{n=1}^{\infty}$ *such that, for every* $n \in \mathbb{N}$

$$H_n = \ln n + c_n$$

STEP 1: The Euler-Mascheroni constant

Let f be a positive, continuous, decreasing function on $[1, \infty)$. Like in the proof of the integral test:

- Sketch the area $A_n = \int_1^n f(x)dx$.
- Draw the lower sum for the partition $\{1, 2, 3, \ldots, n\}$. Call it L_n.
- Let us call $\mu_n = A_n - L_n$. Using the picture, conclude that the sequence $\{\mu_n\}_n^\infty$ is monotonic and bounded. Therefore it is also...?
- Use the above result on the function $f(x) = \frac{1}{x}$ to prove the following:

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> **Theorem** *There exists a convergent sequence $\{c_n\}_{n=1}^\infty$ such that, for every $n \in \mathbb{N}$*
> $$H_n = \ln n + c_n$$

- In particular, this implies that $\boxed{\lim_{n\to\infty} \dfrac{H_n}{\ln n} = 1}$ and that,

$\boxed{\text{for large } n,\ H_n \approx \ln n + \gamma}$, where $\gamma = \lim_{n\to\infty} c_n$ is a new constant.