Test 4 will take place on Friday, March 22, 4:10-6:00pm.

The deadline to request accommodation for a conflict is today, March 15.

Today: Conditional and absolute convergence.

Homework before Monday’s class: watch videos 13.18, 13.19.
Rapid questions: Convergent or divergent?

1. \[ \sum_{n}^{\infty} (1.1)^n \]

2. \[ \sum_{n}^{\infty} (0.9)^n \]

3. \[ \sum_{n}^{\infty} \frac{(-1)^n}{\ln n} \]

4. \[ \sum_{n}^{\infty} \frac{(-1)^n}{e^{1/n}} \]
Rapid questions: Convergent or divergent?

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4. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}} \]

5. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \]

6. \[ \sum_{n=1}^{\infty} \frac{1}{n^{0.9}} \]

7. \[ \sum_{n=1}^{\infty} \frac{n^3 + n^2 + 11}{n^4 + 2n - 3} \]

8. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n^5 + 2n + 16}}{n^4 - 11n + 7} \]
True or False – Absolute Values

1. IF $\{a_n\}_{n=1}^{\infty}$ is convergent, THEN $\{ |a_n| \}_{n=1}^{\infty}$ is convergent.

2. IF $\{ |a_n| \}_{n=1}^{\infty}$ is convergent, THEN $\{ a_n \}_{n=1}^{\infty}$ is convergent.

3. IF $\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\sum_{n=1}^{\infty} |a_n|$ is convergent.

4. IF $\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\sum_{n=1}^{\infty} a_n$ is convergent.
Positive and negative terms

- Let $\sum a_n$ be a series.
- Call $\sum (P.T.)$ the sum of only the positive terms of the same series.
- Call $\sum (N.T.)$ the sum of only the negative terms of the same series.
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Call $\sum (\text{N.T.})$ the sum of only the negative terms of the same series.

<table>
<thead>
<tr>
<th>IF $\sum (\text{P.T.})$ is...</th>
<th>AND $\sum (\text{N.T.})$ is...</th>
<th>THEN $\sum a_n$ may be...</th>
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<tbody>
<tr>
<td>CONV</td>
<td>CONV</td>
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<tr>
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</table>
### Positive and negative terms

- Let $\sum a_n$ be a series.
- Call $\sum (P.T.)$ the sum of only the positive terms of the same series.
- Call $\sum (N.T.)$ the sum of only the negative terms of the same series.

<table>
<thead>
<tr>
<th>$\sum (P.T.)$ may be...</th>
<th>$\sum (N.T.)$ may be...</th>
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<tbody>
<tr>
<td>In general</td>
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<tr>
<td>If $\sum a_n$ is CONV</td>
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<td>If $\sum</td>
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<td>If $\sum a_n$ is ABS CONV</td>
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<td>If $\sum a_n$ is COND CONV</td>
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<td>If $\sum a_n = \infty$</td>
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<td></td>
<td>If $\sum a_n$ is DIV oscillating</td>
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</table>
Your mission: prove ...

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots
\]

\[= \ln 2\]

\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \ldots
\]

\[= \frac{3}{2} \ln 2\]
STEP 1: How quickly do harmonic sums grow?

Let us call \( H_n = \sum_{k=1}^{n} \frac{1}{k} \). It is called a harmonic sum.

This is just notation for a quantity that appears often, like \( n! \):

\[
4! = 4 \cdot 3 \cdot 2 \cdot 1 \quad \quad \quad H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}
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\( \{ H_n \}_{n=1}^{\infty} \) is a new sequence to add to our toolkit, like \( \{ \ln n \}_{n=1}^{\infty} \) or \( \{ n! \}_{n=1}^{\infty} \).
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You are going to prove

**Theorem** There exists a convergent sequence \( \{ c_n \}_{n=1}^{\infty} \) such that, for every \( n \in \mathbb{N} \)

\[
H_n = \ln n + c_n
\]
Let $f$ be a positive, continuous, decreasing function on $[1, \infty)$. Like in the proof of the integral test:

- Sketch the area $A_n = \int_1^n f(x)\,dx$.
- Draw the lower sum for the partition $\{1, 2, 3, \ldots, n\}$. Call it $L_n$.
- Let us call $\mu_n = A_n - L_n$. Using the picture, conclude that the sequence $\{\mu_n\}_n^\infty$ is monotonic and bounded. Therefore it is also...
- Use the above result on the function $f(x) = \frac{1}{x}$ to prove the following:

**Theorem** There exists a convergent sequence $\{c_n\}_{n=1}^\infty$ such that, for every $n \in \mathbb{N}$

$$H_n = \ln n + c_n$$
STEP 1: The Euler-Mascheroni constant

Let $f$ be a positive, continuous, decreasing function on $[1, \infty)$. Like in the proof of the integral test:

- Sketch the area $A_n = \int_1^n f(x) \, dx$.
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- Let us call $\mu_n = A_n - L_n$. Using the picture, conclude that the sequence $\{\mu_n\}_n$ is monotonic and bounded. Therefore it is also...
- Use the above result on the function $f(x) = \frac{1}{x}$ to prove the following:

**Theorem** There exists a convergent sequence $\{c_n\}_{n=1}^{\infty}$ such that, for every $n \in \mathbb{N}$

$$H_n = \ln n + c_n$$

- In particular, this implies that $\lim_{n \to \infty} \frac{H_n}{\ln n} = 1$ and that, for large $n$, $H_n \approx \ln n + \gamma$, where $\gamma = \lim c_n$ is a new constant.