Today: Suprema and infima.

Homework before Friday’s class: watch videos 7.5, 7.6, 7.7, 7.8, 7.9.
Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, 0) \cup (3, \infty)$
3. $C = \{2, 3, 4\}$
4. $D = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n > 0 \right\}$
5. $E = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{Z}, n > 0 \right\}$
6. $F = \{2^n : n \in \mathbb{Z}\}$
Empty set

1. Does $\emptyset$ have an upper bound?
2. Does $\emptyset$ have a supremum?
3. Does $\emptyset$ have a maximum?
4. Is $\emptyset$ bounded above?

Recall:
Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

$a$ is an upper bound of $A$ means $\forall x \in A$, $x \leq a$.

$a$ is the least upper bound (lub) or supremum (sup) of $A$ means $a$ is an upper bound of $A$, and there are no smaller upper bounds.
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Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- $a$ is an upper bound of $A$ means $\forall x \in A, x \leq a$.
- $a$ is the least upper bound (lub) or supremum (sup) of $A$ means
  - $a$ is an upper bound of $A$, and
  - there are no smaller upper bounds.
Equivalent definitions of supremum

Assume $S$ is an upper bound of the set $A$. Which one (or ones) of the following statements is equivalent to “$S$ is the supremum of $A$”?

1. If $R$ is an upper bound of $A$, then $S \leq R$.
2. $\forall R \geq S$, $R$ is an upper bound of $A$.
3. $\forall R \leq S$, $R$ is not an upper bound of $A$.
4. $\forall R < S$, $R$ is not an upper bound of $A$.
5. $\forall R < S$, $\exists x \in A$ such that $R < x$.
6. $\forall R < S$, $\exists x \in A$ such that $R \leq x$.
7. $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.
8. $\forall R < S$, $\exists x \in A$ such that $R < x < S$.
9. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$. 
Infima and suprema exercises

Let $A, B \subseteq \mathbb{R}$. Which of the following are true or false? If false, find a counterexample.

1. If $B \subseteq A$ and $A$ is bounded above, then $B$ is bounded above.
2. If $B \subseteq A$ and $B$ is bounded above, then $A$ is bounded above.
3. If $B \subseteq A$ and $A$ is bounded above, then $\sup B \leq \sup A$.
4. If $B \subseteq A$ and $A$ is bounded above, then $\inf B \leq \inf A$.
5. If $A$ and $B$ are bounded above and $\sup A \leq \sup B$, then $A \subseteq B$.
6. If $A$ and $B$ are bounded above, then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
7. If $A$ and $B$ are bounded above, then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
Let $f(x) = x^3 - 3x$.
Find four open intervals $I_1, I_2, I_3, I_4$ such that

1. $f$ has a maximum and a minimum on $I_1$.
2. $f$ has a supremum and no maximum on $I_2$.
3. $f$ has a supremum and no infimum on $I_3$.
4. $f$ does not have a supremum or an infimum on $I_4$. 
1. Let $f$ and $g$ be bounded functions on $[a, b]$. Then

$$\sup_{[a, b]} (f + g) = \sup_{[a, b]} f + \sup_{[a, b]} g$$

2. Let $a < b < c$. Let $f$ be a bounded function on $[a, c]$. Then

$$\sup_{[a, c]} f = \sup_{[a, b]} f + \sup_{[b, c]} f$$

3. Let $f$ be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\sup_{[a, b]} (cf) = c \left( \sup_{[a, b]} f \right)$$