- Today: Antiderivatives and indefinite integrals.
- Homework before Tuesday's class: watch videos 8.3, 8.4.
- Test 3 will take place on Friday, February 10, 4-6pm, see details on Quercus.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \quad \int_0^4 f(x)dx = 9, \quad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$ $= 3$
2. $\int_0^2 f(t)dx$ $= 2f(t)$
3. $\int_2^0 f(x)dx$ $= -\int_0^2 f(x)dx = -3$
4. $\int_2^4 f(x)dx$ $= \int_0^4 - \int_0^2 = 9-3=6$
5. $\int_{-2}^0 f(x)dx$?
6. $\int_0^4 [f(x) - 2g(x)] \; dx$

Find a function f such that

- For every $x \in \mathbb{R}$, $f''(x) = \sin x + x^2$,
- $f'(0) = 5$,
- $f(0) = 7$.

$$f'(x) = -\cos x + \frac{x^3}{3} + C$$

$f'(0) = 5$ $\qquad -\cos 0 + \frac{0^3}{3} + C = 5$

$$-1+0+C=5 \Rightarrow C=6$$

$$\Rightarrow f'(x) = -\cos x + \frac{x^3}{3} + 6$$

Then $f(x) = -\sin x + \dfrac{x^4}{4\cdot 3} + 6x + B$

$$f(0)=7 \Rightarrow -\sin 0 + \frac{0^4}{12} + 6\cdot 0 + B = 7$$

$$\Rightarrow B=7 \Rightarrow \boxed{f(x) = -\sin x + \frac{x^4}{12} + 6x + 7}$$

Functions defined by integrals

Which ones of these are valid ways to define functions?

$\cancel{\sum_{i=1}^{i} a_i}$

✓ 1. $F(x) = \int_0^x \frac{t}{1+t^8}\, dt$

✗ 2. $F(x) = \int_0^x \frac{x}{1+x^8}\, dx$

✓ 3. $F(x) = \int_0^x \frac{x}{1+t^8}\, dt$

✓ 4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\, dt$

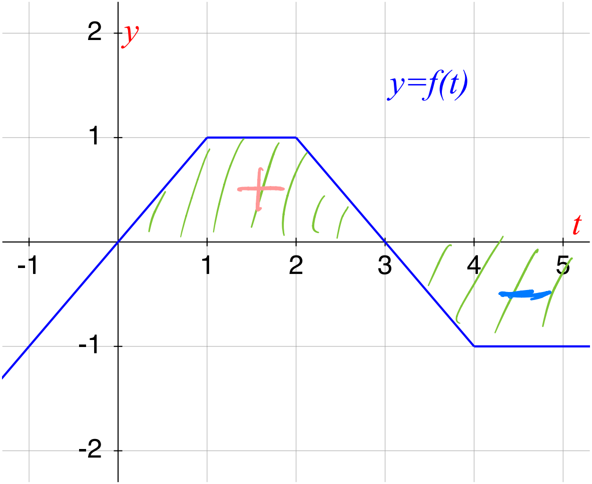
✓ 5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\, dt$

✓ 6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\, dt$

✓ 7. $F(x) = x \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$

✗ 8. $F(x) = t \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$

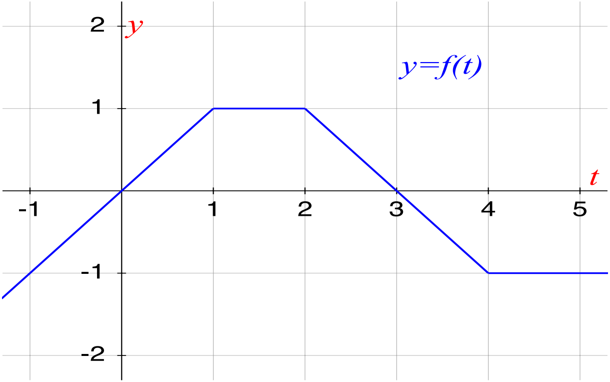
Towards FTC



Compute:

1. $\int_0^1 f(t)dt = \frac{1}{2}$
2. $\int_0^2 f(t)dt = 1\frac{1}{2}$
3. $\int_0^3 f(t)dt = 2$
4. $\int_0^4 f(t)dt = 1\frac{1}{2}$
5. $\int_0^5 f(t)dt = \frac{1}{2}$

Towards FTC (continued)

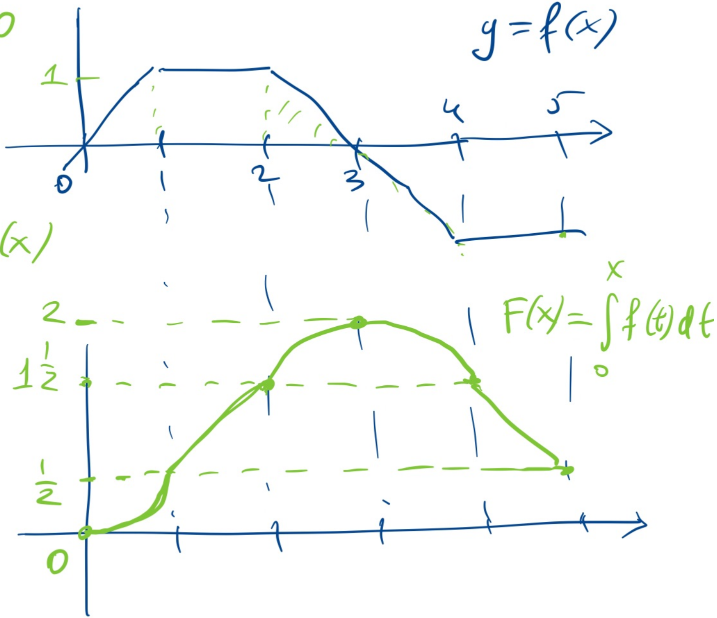


Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Note: $F(0)=0$

$F'(x)=f(x)$



Antiderivatives

Compute these antiderivatives by guess-and-check. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$n \neq -1$

$n \in \mathbb{R}$

✓ 1. $\int x^5 dx = \frac{x^6}{6} + C$

2. $\int \left(3x^8 - 18x^5 + x + 1\right) dx$

3. $\int \sqrt[3]{x}\, dx$

4. $\int \frac{1}{x^9}\, dx$

✓ 5. $\int \sqrt{x}\left(x^2 + 5\right) dx$ $= \int (x^{2.5} + 5x^{0.5}) dx$

6. $\int \frac{1}{e^{2x}}\, dx$

7. $\int \sin(3x)\, dx$

8. $\int \cos(3x + 2)\, dx$

✓ 9. $\int \sec^2 x\, dx$

✓ 10. $\int \sec x \tan x\, dx$

11. $\int \frac{1}{x}\, dx$

12. $\int \frac{1}{x + 3}\, dx$

9. $$\int \frac{dx}{\cos^2 x} = \tan x + C \qquad \forall C \in \mathbb{R}$$

10. $$\int \sec x \tan x \, dx = \int \frac{\sin x}{\cos^2 x} dx = -\int \frac{d(\cos x)}{\cos^2 x}$$

$$\underset{\cos x = t}{=} -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{\cos x} + C$$

Check $$\left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x}$$

$$d(\cos x) = \underbrace{(\cos x)'}_{-\sin x} dx$$

1. Calculate

$$\frac{d}{dx}\left[e^x \sin x\right], \qquad \frac{d}{dx}\left[e^x \cos x\right].$$

2. Use the previous answer to calculate

$$\int e^x \sin x \; dx, \qquad \int e^x \cos x \; dx.$$

$$(e^x \sin x)' = e^x \sin x + e^x \cos x$$

$$\pm \quad (e^x \cos x)' = -e^x \sin x \quad e^x \cos x$$

$$\left(e^x(\sin x + \cos x)\right)' = 2e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

$$\left(e^x(\sin x - \cos x)\right)' = 2e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

A harder antiderivative

1. Calculate

$$\frac{d}{dx}\left[\arctan x\right], \qquad \frac{d}{dx}\left[\frac{x}{1+x^2}\right].$$

2. Use the previous answer to calculate

$$\int \frac{1}{\left(1+x^2\right)^2}\, dx$$