Examples and properties of the integral.

Homework before Wednesday’s class: watch videos 7.9, 7.10.
Lower sums, upper sums, and integrability: illustration

\[ \underline{L}_a^b(f) \leq \overline{L}_a^b(f) \]
Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute all the upper sums and all the lower sums.
3. Compute $L_0^4(f)$
4. Compute $\overline{L}_0^4(f)$
5. Is $f$ integrable on $[0, 4]$?
Example 2: a non-continuous function

Consider the function \( f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases} \), defined on \([0, 1]\).

1. Let \( P = \{0, 0.2, 0.5, 0.9, 1\} \).
   Calculate \( L_P(f) \) and \( U_P(f) \) for this partition.

2. Fix an arbitrary partition \( P = \{x_0, x_1, \ldots, x_N\} \) of \([0, 1]\).
   What is \( U_P(f) \)? What is \( L_P(f) \)? (Draw a picture!)

3. Find a partition \( P \) such that \( L_P(f) = 4.99 \).

4. What is the upper integral, \( \overline{I}_0^1(f) \)?

5. What is the lower integral, \( \underline{I}_0^1(f) \)?

6. Is \( f \) integrable on \([0, 1]\)?
Example 3: a very non-continuous function

Consider the function defined on $[0, 1]$:

$$f(x) = \begin{cases} 
1/2 & \text{if } 0 \leq x \leq 1/2 \\
1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\
0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q}
\end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition $P$ such that $L_P(f) = \frac{1}{4}$ and $U_P(f) = \frac{3}{4}$
4. What is the upper integral, $\overline{I}_0^1(f)$?
5. What is the lower integral, $\underline{I}_0^1(f)$?
6. Is $f$ integrable on $[0, 1]$?
Is this possible?

Find bounded functions $f$ and $g$ on $[0, 1]$ such that

- $f$ is not integrable on $[0, 1]$,
- $g$ is not integrable on $[0, 1]$,
- $f + g$ is integrable on $[0, 1]$.

or prove this is impossible.
Properties of the integral

Assume we know the following

\[\int_0^2 f(x)\,dx = 3, \quad \int_0^4 f(x)\,dx = 9, \quad \int_0^4 g(x)\,dx = 2.\]

Compute:

1. \[\int_0^2 f(t)\,dt\]
2. \[\int_0^2 f(t)\,dx\]
3. \[\int_0^2 f(x)\,dx\]
4. \[\int_2^4 f(x)\,dx\]
5. \[\int_{-2}^0 f(x)\,dx\]
6. \[\int_0^4 [f(x) - 2g(x)]\,dx\]