• Examples and properties of the integral.

• Homework before Wednesday’s class: watch videos 7.9, 7.10.

• Test 3 is on Feb 3. It covers Ch 6, 7, 8.
  Check the info on Quercus
Lower sums, upper sums, and integrability: illustration

\[ I_a^b(f) \leq \overline{I}_a^b(f) \]

finer partitions

lower sums

upper sums

finer partitions
Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.  

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute all the upper sums and all the lower sums.
3. Compute $I^4_0(f) = 8$
4. Compute $\overline{I}^4_0(f) = 8$
5. Is $f$ integrable on $[0, 4]$? yes
\[ \Delta x_i = x_i - x_{i-1}, \]

\[ m_i = \inf_{x_N} f = \min_{x_N} f = 2 \]

\[ M_i = \sup_{x_N} f = \max_{x_N} f = 2 \]

\[ L_p(f) = \sum_{i=1}^{N} m_i \Delta x_i = \sum_{i=1}^{N} 2(x_i - x_{i-1}) \]

\[ = 2(x_N - x_0) = 2(b - a) = 2 \cdot 4 = 8 \]

\[ U_p(f) = \sum M_i \Delta x_i = 2 \cdot 4 = 8 \]

for any \( P \) partition of \([0,4]\)

\[ \int_{[0,4]} f = \lim_{\mathcal{P} \to \infty} \sup P L_p(f) = \sup P \inf \longleftarrow \int_0^4 f(x) \]
Example 2: a non-continuous function

Consider the function \( f(x) = \begin{cases} 
0 & x = 0 \\
5 & 0 < x \leq 1 
\end{cases} \), defined on \([0, 1]\).

1. Let \( P = \{0, 0.2, 0.5, 0.9, 1\} \).
   Calculate \( L_P(f) \) and \( U_P(f) \) for this partition.

2. Fix an arbitrary partition \( P = \{x_0, x_1, \ldots, x_N\} \) of \([0, 1]\).
   What is \( U_P(f) \)? What is \( L_P(f) \)? (Draw a picture!)

3. Find a partition \( P \) such that \( L_P(f) = 4.99 \).

4. What is the upper integral, \( \overline{I}_0^1(f) \)?

5. What is the lower integral, \( \underline{I}_0^1(f) \)?

6. Is \( f \) integrable on \([0, 1]\)? yes
\( L_p (f) = 0.2 + 5(0.3 + 0.4 + 0.1) \)
\[= 5 \cdot 0.8 = 4 \]

\( U_p (f) = 5 \cdot (0.2 + 0.3 + 0.4 + 0.1) \)
\[= 5 \cdot 1 = 5 \]

For any partition \( P = \{x_0, \ldots, x_N\} \)
\( U_p (f) = \sum \left( \sup_{x_i} f \right) \Delta x_i = \sum 5 \cdot \Delta x_i = 5 \cdot (b-a) \)
\[= 5 \cdot 1 = 5 \]

\[\inf_P U_p = 5 = \overline{I}_D (f)\]
\[ \inf t = 5 \quad \text{for} \quad i = 2, \ldots, N, \quad \text{but} \]
\[ \Delta x_i \]
\[ \inf t = 0. \quad \text{Hence} \]
\[ \Delta x_i \]
\[ L_p(t) = 0 \cdot \Delta x_1 + 5(\Delta x_2 + \cdots + \Delta x_N) = 5(1 - x_1) \]
\[ x_2 - x_1 \quad x_3 - x_2 \quad x_N - x_{N-1} \]
\[ \text{Set} \quad 5(1 - x_1) = 4.99 \quad \Rightarrow \quad 5 - 5x_1 = 5 - 0.01 \]
\[ x_1 = 0.01 \]
\[ 5x_1 = 0.01 \]
\[ \exists \, \text{a take} \quad P = \{0, 0.002, 1\} \]
\[ L_p(t) = 4.99 \]
\[ \sup_{x_i \in (0, 1)} L_p(t) = \sup_{x_i \in (0, 1)} 5(1 - x_i) = 5 = \overline{I}_0 = \overline{I}_0 \]
Example 3: a very non-continuous function

Consider the function defined on $[0, 1]$:

$$f(x) = \begin{cases} 
1/2 & \text{if } 0 \leq x \leq 1/2 \\
1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\
0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q}
\end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition $P$ such that $L_P(f) = \frac{1}{4}$ and $U_P(f) = \frac{3}{4}$.

4. What is the upper integral, $\overline{\int_0^1}(f)$? $\frac{3}{4}$
5. What is the lower integral, $\underline{\int_0^1}(f)$? $\frac{1}{4}$
6. Is $f$ integrable on $[0, 1]$? *No*
\[ L_p(f) = \frac{1}{2} \cdot 0.2 + \frac{1}{2} \cdot 0.2 + 0.0.2 + 0.0.2 + 0.0.2 \]
\[ \inf f = \frac{1}{2} = \sup f \]
\[ \text{in } [0, 0.2] \]
\[ \text{in } [0.02] \]
\[ \inf f = 0 \text{ since } \]
\[ \text{in } [0.4, 0.6] \]
\[ \exists \text{ inrat- } f \in [0.4, 0.6] \]

\[ L_p = \frac{1}{2} (0.2 + 0.2) = \frac{0.4}{2} = 0.2 \]

\[ U_p = \frac{1}{2} \cdot 0.2 + \frac{1}{2} \cdot 0.2 + 1 \cdot 0.2 + 1 \cdot 0.2 + 1 \cdot 0.2 \]
\[ = \frac{1}{2} \cdot 0.4 + 1 \cdot 0.6 = 0.2 + 0.6 = 0.8 \]
Take $P = \{0, \frac{1}{2}, 1\}$

$$L_p(f) = \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{4}$$

$$U_p(f) = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

For any partition not containing point $\frac{1}{2}$
Find bounded functions $f$ and $g$ on $[0,1]$ such that

- $f$ is not integrable on $[0,1]$,
- $g$ is not integrable on $[0,1]$,
- $f + g$ is integrable on $[0,1]$.

or prove this is impossible.