- Today: Integral as limits.

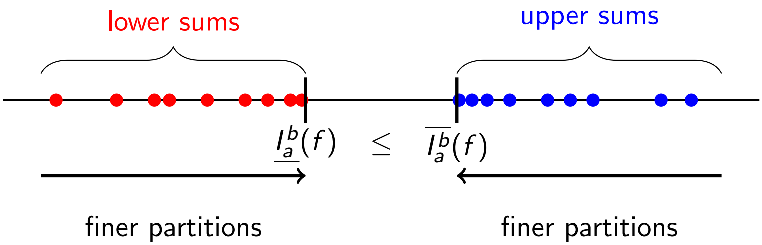
- Homework before Thursday's class: watch videos 8.1, 8.2.

The "ε–characterization" of integrability

True or False?

Let f be a bounded function on $[a, b]$.

1. IF f is integrable on $[a, b]$
   THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
2. IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$",
   THEN f is integrable on $[a, b]$



The "ε–characterization" of integrability - First proof

You are going to prove

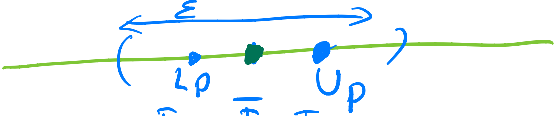
Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

1. Recall the definition of "f is integrable on $[a, b]$".
2. Write down the structure of the proof.
3. Fix $\varepsilon > 0$. Show there is a partition P s.t. $U_P(f) < \overline{I_a^b}(f) + \dfrac{\varepsilon}{2}$. Why?
4. Assume f is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions P_1 and P_2 s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.
6. Write down a proof for the claim.

Pf



f is integ $\Leftrightarrow$ $\underline{I} = \overline{I} = I$

fix $\forall \varepsilon > 0$ $\overline{I} = \inf\limits_{P} U_P \Rightarrow \exists P_1$ s.t.

$$\overline{I} \le U_{P_1} < \overline{I} + \frac{\varepsilon}{2}$$

Similarly, $\underline{I} = \sup\limits_{P} L_P \Rightarrow \exists P_2$ s.t.

$$\underline{I} - \frac{\varepsilon}{2} < L_{P_2} \le \underline{I}$$

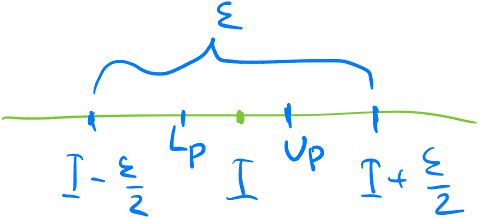
Set $P = P_1 \cup P_2$ $\leftarrow$ finer than both P_1 and P_2

$$\underline{I} - \frac{\varepsilon}{2} < L_{P_2} \le L_P \le U_P \le U_{P_1} < \overline{I} + \frac{\varepsilon}{2}$$

$$\underline{I} = I, \qquad \overline{I} = I$$

$$\Rightarrow \quad U_P - L_P < \varepsilon$$

$\blacksquare$



The "ε–characterization" of integrability - Second proof

You are going to prove

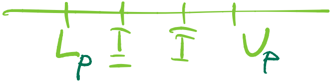
Claim

Let f be a bounded function on $[a, b]$.

- IF "$\forall\varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
- THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.
4. Write down a proof for the claim.

Pf sketch $\forall P$ $\quad L_P \le \underline{I} \le \overline{I} \le U_P$



$\forall \varepsilon > 0 \quad \exists P$ s.t.

$$0 \le \overline{I} - \underline{I} \le U_P - L_P < \varepsilon \Rightarrow \overline{I} = \underline{I}$$

$\Rightarrow f$ is integr. $\blacksquare$

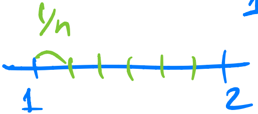
The norm of a partition

$||P|| = \max_i |\Delta x_i|$

$\nexists$ P like this

1. Construct a partition P of $[1, 2]$ such that $||P|| = \frac{5}{\pi}$. $\checkmark$ 1
2. Construct a sequence of partitions

$$P_1, P_2, P_3, \ldots$$



of $[1, 2]$, *as simple as possible*, such that $\lim_{n\to\infty} ||P_n|| = 0$.

3. Construct a *different* sequence of partitions

$$Q_1, Q_2, Q_3, \ldots$$

of $[1, 2]$, such that $\lim_{n\to\infty} ||Q_n|| = 0$.

2) $P_n = \left\{ 1 + \frac{i}{n} \;\middle|\; i = 0, \dots, n \right\}$

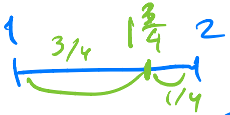
$$\parallel$$

$$\left\{ 1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, \underbrace{1 + \frac{n}{n}}_{=2} \right\}$$

$\|P_n\| = \frac{1}{n}$, $\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{1}{n} = 0$

3) $Q_n = P_n \cup \left\{ \frac{\pi}{e} \right\}$

$Q_n = \left\{ 1 + \frac{i}{2^n} \;\middle|\; i = 0, \dots, 2^n \right\}$



$$\|P\| = \max\left\{\frac{1}{4}, \frac{3}{4}\right\} = \frac{3}{4}$$

$$P = \left\{1, 1\frac{3}{4}, 2\right\}$$

Find P s.t. $\|P\| = \frac{3}{100}$

Take $P = \{1, 1.03, 1.06, \ldots, 1.99, 2\}$

Riemann sums example

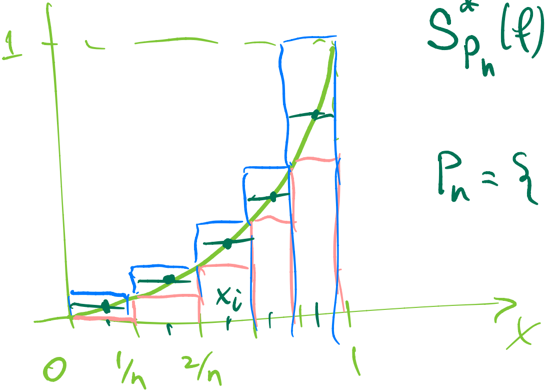
Imitate the calculation in Video 7.11.

Exercise

Let $f(x) = x^2$ on $[0, 1]$.
Let $P_n = \{\text{breaking the interval into } n \text{ equal pieces}\}$.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write $S^*_{P_n}(f)$ as a sum when we choose x_i^* as the right end-point.
4. Add the sum
5. Compute $\lim\limits_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x_i^* as the left end-point.

Helpful formulas: $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$



$$S^*_{P_n}(f) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$$P_n = \left\{ \frac{k}{n},\ k = 0, \dots, n \right\}$$

$$|\Delta x_k| = \frac{1}{n} \quad \forall k$$

$$S^*_{P_n}(f) = \sum_{i=1}^{n} \underset{\underset{\left(\frac{i}{n}\right)^2}{\parallel}}{f(x_i)}\ \underset{\underset{\frac{1}{n}}{\parallel}}{\Delta x_i} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6 \cdot n^3} \underset{n\to\infty}{\approx} \frac{2n^3}{6n^3} = \frac{1}{3}$$

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\tilde{S}^x_{P_n}(f) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (i^2 - 2i + 1) \xrightarrow[n\to\infty]{} \frac{2}{6} = \frac{1}{3}$$