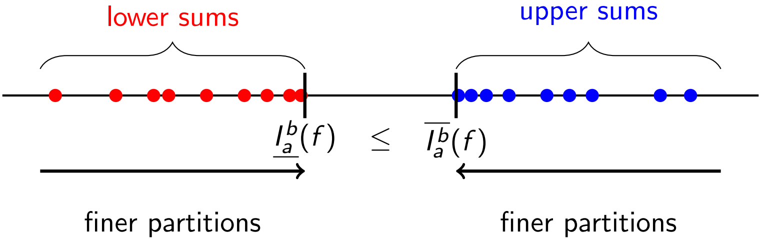
- Examples and properties of the integral.

- Homework before Wednesday's class: watch videos 7.9, 7.10.

Lower sums, upper sums, and integrability: illustration



Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute *all* the upper sums and *all* the lower sums.
3. Compute $\underline{I_0^4}(f)$
4. Compute $\overline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Example 2: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

Example 3: a very non-continuous function

Consider the function defined on $[0,1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition P such that $L_P(f) = \frac{1}{4}$ and $U_P(f) = \frac{3}{4}$
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0,1]$?

Is this possible?

Find bounded functions f and g on $[0,1]$ such that

- f is not integrable on $[0,1]$,
- g is not integrable on $[0,1]$,
- $f+g$ is integrable on $[0,1]$.

or prove this is impossible.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \quad \int_0^4 f(x)dx = 9, \quad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(t)dx$
3. $\int_2^0 f(x)dx$
4. $\int_2^4 f(x)dx$
5. $\int_{-2}^0 f(x)dx$
6. $\int_0^4 [f(x) - 2g(x)] \; dx$